Semistate models of electrical circuits including memristors*

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Abstract

In this paper we present several semistate or differential-algebraic models arising in nodal analysis of nonlinear circuits including memristors. The goal is to characterize the tractability index of these models under strict passivity assumptions, a key issue for the numerical simulation of circuit dynamics. We show that the main model, which combines memristors' fluxes and charges, is index two. From a technical point of view, this result is based on the use of a projector along the image of the leading matrix, in contrast to previous index analyses. For charge-controlled memristors, the elimination of fluxes yields an index one system in topologically nondegenerate circuits, and an index two model otherwise. Analogous results are also proved to hold for flux-controlled memristors. Our framework accommodates coupling effects among resistors, memristors, capacitors and inductors.

Keywords: memristor, lumped circuit, nodal analysis, differential-algebraic equation, semistate model, index.

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1 Introduction

In 1971, Leon Chua introduced the scientific foundation for a new circuit element named the *memory resistor* or *memristor* [11]. This element would be defined by a nonlinear relationship between the charge and the magnetic flux. Such a physical device would not be reported until 2008, when a physical model of a two-terminal device behaving as a memristor was announced in *Nature* [45].

This discovery has attracted a great attention. Many potential applications are reported, and several papers have been already published on this topic, specially at the device modeling level; see e.g. [3, 10, 44, 50, 51]. As indicated in [45], memristance becomes important for understanding the electronic characteristics of devices at the nanometer scale. Memristors can also be used to design nonlinear oscillators by letting them be the nonlinear device in Chua's circuit family [25].

The systematic use of memristors within electrical and electronic engineering, specially in large scale circuits, will require some effort at the circuit modeling level. The nonlinear nature of the memristor necessarily leads to time-domain models. In this context, it is worth remarking that applications such as the ones discussed in [25] are based on state space models. Note that the formulation of these models is feasible because of the low number of devices used in the circuits. However, in integrated electronics, where millions of devices are involved, the formulation of state models is not easily automatable. Instead, *semistate models* based on differential-algebraic equations (DAEs) are preferred [14, 38, 39, 40, 41, 46, 49], following the seminal work of Dziurla and Newcomb [13, 34]. Actually, nodal analysis methods such as the ones used in SPICE or in commercial variants as HSPICE (Synopsys) and PSPICE (Cadence) set up the circuit equations in differential-algebraic form: cf. Section 2 and [15, 20, 21, 32, 47, 48].

A major problem in DAE modeling of electrical and electronic circuits is the characterization of the model *index*. The index can be defined in several interrelated ways; see Section 3 below, and [5, 18, 23, 28, 36, 42] for a discussion of different index notions which generalize the Kronecker index of a matrix pencil [17]. Broadly speaking, the index can be seen as a measure of the system sensitivity to input perturbations, and also as a measure of the difficulties to be found in the numerical simulation of the system dynamics. In the strictly passive context, the index of different DAE models of circuits composed of resistors, inductors, capacitors, and voltage and current sources is known to be no greater than two [15, 20, 21, 40, 42, 43, 47, 48].

In the present paper we extend these index analyses to strictly passive circuits including memristors. As in previous works [15, 43, 47, 48] the mathematical discussion is based on the projector-based tractability index framework; however, the presence of memristors introduces technical difficulties which can be overcome by using a projector along the image of the leading matrix, in contrast to previous approaches. We will show that, under both charge- and flux-control assumptions on memristors, models involving both the flux and the charge are always index two, whereas the elimination of one of these variables (defined by the control assumption on memristors) may lead to index one or index two models, depending on the circuit topology. These results are detailed in Section 4. An example and some concluding remarks can be found in Sections 5 and 6, respectively.

2 Background

2.1 The memristor

A *memristor* or *memory resistor* [11] is an electrical device governed by a nonlinear flux-charge relation of the type

$$g(\varphi, q) = 0. \tag{1}$$

If we assume the device to be charge-controlled, this characteristic takes the form

$$\varphi = \phi(q). \tag{2}$$

Provided that ϕ is a differentiable mapping (in the sequel, we will assume all the mappings to be differentiable enough without further explicit mention), the incremental *memristance* is defined as

$$M(q) = \phi'(q) = \frac{d\phi(q)}{dq}. \tag{3}$$

The standard electromagnetic relations $\varphi'(t) = v_m(t)$, $q'(t) = i_m(t)$ yield $v_m(t) = M(q(t))i_m(t)$, where v_m and i_m are the voltage across and the current through the memristor, respectively. The device behaves as a resistor in which the resistance depends on $q(t) = \int_{-\infty}^{t} i_m(\tau)d\tau$, hence the name. It is an essentially nonlinear device, since a linear characteristic (2) would make M constant and there would be no difference to a resistor.

If the device is flux-controlled, then the characteristic (1) reads

$$q = \sigma(\varphi) \tag{4}$$

and the incremental *memductance* is

$$W(\varphi) = \sigma'(\varphi) = \frac{d\sigma(\varphi)}{d\varphi}. \tag{5}$$

If both the charge-controlled and the flux-controlled descriptions are well-defined (maybe in a local sense), then because of the inverse function theorem the memristance and the memductance are inverse to each other, that is, $(W(\varphi))^{-1} = M(\sigma(\varphi))$.

We will say that a memristor is *strictly passive* if $M(q) > 0$ or $W(\varphi) > 0$, respectively. It will be called *passive* if $M(q) \geq 0$ or $W(\varphi) \geq 0$. These assumptions may be understood to hold globally (that is, for all q or all φ) or locally around a given operating point.

Certainly, the circuits under study may include more than one memristor. Provided that there are m memristors, then ϕ and σ must be understood as mappings $\mathbb{R}^m \to \mathbb{R}^m$, and the memristance and memductance will be the corresponding Jacobian matrices, lying on $\mathbb{R}^{m\times m}$. Even though we are not yet aware of applications exhibiting coupling effects among memristors, from a mathematical point of view we find no difficulty in accommodating coupled memristors in our framework, just by allowing the memristance and memductance matrices to be non-diagonal. In this context, the strict passivity (resp. the passivity) condition requires M or W to be positive definite (resp. positive semidefinite), that is, to verify $u^T M(q)u > 0$ or $u^T W(\varphi)u > 0$ (resp. ≥ 0) for any $u \in \mathbb{R}^m - \{0\}$. For later use, keep in mind that a non-singular matrix is positive definite if and only if so it is its inverse; namely, M will be positive definite if and only if so it is W, and the same will happen with the inverse conductance, capacitance and inductance matrices G^{-1}, C^{-1} and L^{-1}. In an uncoupled setting, the strict passivity condition amounts to requiring that the incremental memristances (or memductances) of the individual devices are positive.

2.2 Semistate circuit models arising in nodal analysis

For the sake of simplicity we will assume that capacitors and inductors are voltage/current controlled through certain mappings $q_c = \psi(v_c)$, $\varphi_l = \eta(i_l)$, respectively. This way we may eliminate capacitor charges and inductor fluxes from the equations, making the discussion easier. We denote by $C(v_c)$ and $L(i_l)$ the incremental capacitance and inductance matrices $\psi'(v_c)$, $\eta'(i_l)$.

The nodal equations for a circuit composed of resistors, memristors, inductors, capacitors and voltage and current sources can be then written in the following differential-algebraic form:

$$\begin{aligned}
C(v_c)v_c' &= i_c && \text{(6a)}\\
L(i_l)i_l' &= A_l^T e && \text{(6b)}\\
\varphi_m' &= A_m^T e && \text{(6c)}\\
q_m' &= i_m && \text{(6d)}\\
0 &= A_r\gamma(A_r^T e) + A_c i_c + A_l i_l + A_m i_m + A_v i_v + A_i i_s(t) && \text{(6e)}\\
0 &= v_c - A_c^T e && \text{(6f)}\\
0 &= v_s(t) - A_v^T e && \text{(6g)}\\
0 &= g(\varphi_m, q_m). && \text{(6h)}
\end{aligned}$$

Here, i, v, q and φ stand for branch currents, voltages, charges and fluxes, whereas e represents node voltages. The subindices c, l, m, r, v, i in branch variables stand for capacitors, inductors, memristors, resistors, and voltage and current sources. Equation (6e) expresses Kirchhoff's current law in the form $Ai = 0$, using the reduced incidence matrix A defined in subsection 2.3 below; note that we are assuming resistors to be voltage-controlled by the relation $i_r = \gamma(v_r)$. In turn, Kirchhoff's voltage law is used in the form $v = A^T e$ to express the voltages in inductors, memristors, resistors, capacitors and voltage sources in terms of e within equations (6b), (6c), (6e), (6f) and (6g), respectively. Equation (6h) describes the constitutive relation of memristors; for the moment we do not make any assumption on the controlling variables for them, and (6h) may accommodate both charge-controlled and flux-controlled memristors. The excitation terms $v_s(t)$, $i_s(t)$ come from the voltage and current sources, respectively; we assume that all sources are independent, although the results can be extended to a broad class of controlled sources as in [15, 42]. For later use, we define the incremental conductance as $G(A_r^T e) = \gamma'(A_r^T e)$.

2.3 Digraphs

Many properties of an electrical circuit can be expressed in terms of its underlying directed graph or *digraph*. We compile below, for later use, some elementary notions coming from digraph theory: details can be found e.g. in [1, 4, 12, 16, 42].

Let n and b stand for the number of circuit nodes and branches. After choosing a reference node, the reduced incidence matrix $A \in \mathbb{R}^{(n-1)\times b}$ is defined as (a_{ij}) with

$$a_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ is not incident with node } i. \end{cases}$$

Provided that the circuit is connected, the reduced incidence matrix has maximal row rank (cf. [1, p. 145] or [16, p. 78]). The matrix A is split as $(A_r\ A_c\ A_l\ A_m\ A_v\ A_i)$, where A_r (resp. A_c, A_l, A_m, A_v, A_i) describes the incidence between resistive (resp. capacitive, inductive, memristive, voltage source, current source) branches and nodes.

A key role in our analysis will be played by certain types of loops and cutsets. A subset $\mathcal{K}$ of the set of branches of a connected digraph is a *cutset* if the deletion of $\mathcal{K}$ results in a disconnected digraph, and it is minimal with respect to this property (namely, the deletion of any proper subset of $\mathcal{K}$ does not disconnect the digraph). Loops and cutsets defined by specific types of branches can be characterized in terms of the incidence matrix, as stated in Lemmas 1 and 2 below. Both are standard results in graph theory (see e.g. [1, 42]). We denote by $A_{\mathcal{K}}$ (resp. $A_{\mathcal{G}-\mathcal{K}}$) the submatrix of the reduced incidence matrix formed by the columns defined by the branches in $\mathcal{K}$ (resp. not in $\mathcal{K}$).

Lemma 1. *Let $\mathcal{K}$ be a subset of the set of branches of a connected digraph $\mathcal{G}$. $\mathcal{K}$ does not contain cutsets if and only if* Ker $A_{\mathcal{G}-\mathcal{K}}^T = \{0\}$*, that is, iff $x^T A_{\mathcal{G}-\mathcal{K}} = 0 \Rightarrow x = 0$.*

Lemma 2. *Let $\mathcal{K}$ be a subset of the set of branches of a connected digraph $\mathcal{G}$. $\mathcal{K}$ does not contain loops if and only if* Ker $A_{\mathcal{K}} = \{0\}$*, that is, iff $A_{\mathcal{K}} y = 0 \Rightarrow y = 0$.*

3 Semistate circuit models and their index

System (6) is a quasilinear differential-algebraic equation (DAE) of the form

$$E(x)x' = f(x) + s(t), \tag{7}$$

where x groups together all the circuit variables entering (6); the right-hand side can be written in this form after distinguishing the excitation terms $s(t)$ coming from the voltage and current sources. When the leading matrix $E(x)$ is singular, some of the variables entering

(7) become redundant and this makes it a *semistate system*. We present in this Section a brief introduction to semistate systems and DAEs. The attention will be mainly focused on their *index*, a concept which articulates many analytical and numerical properties of semistate equations.

3.1 Semistate equations

Quasilinear DAEs arise in different application fields, including not only circuit theory but also mechanics, control theory, chemical processes, power systems, etc. Detailed introductions to DAE theory and their applications can be found in [5, 18, 23, 28, 36, 42]. In the electrical circuit context, DAEs are usually referred to as *semistate equations*, this name stemming from the original work of Dziurla and Newcomb [13, 34].

DAEs became the object of increasing interest in the 1980s; see for instance the classical "Pitman books" by Campbell [6, 7], the first edition of [5], the seminal work on projector methods [18] or the 1989 special issue of *Circuits, Systems, and Signal Processing*. This research led, in the 1990s, to different mathematical frameworks for the analysis of DAEs, structured around different *index* notions; these notions include the differentiation [5, 8, 9], tractability [19, 29], geometric [35, 37], perturbation [22, 23], and strangeness [26, 27] indices. These rigorous mathematical foundations have led, in the last decade, to a systematic application of DAE theory in different fields; cf. in particular [14, 15, 20, 21, 32, 38, 40, 43, 46, 47, 48] as a sample of works on semistate modeling of lumped circuits.

The different index notions can be seen as an extension to time-varying and/or nonlinear contexts of the Kronecker index (or nilpotency index) of a matrix pencil [17]. In a time-invariant setting, the different index concepts equal the Kronecker index, but in time-varying or nonlinear problems they may be different; moreover, the aforementioned frameworks differ in the way in which the index is computed and the kind of solvability results which are supported on them.

In particular, the *differentiation index* can be roughly defined as the number of differentiations needed to get an explicit ODE describing the DAE solutions. Consider, for example, a *semiexplicit* DAE

$$y' = f(y,z) \tag{8a}$$
$$0 = g(y,z) \tag{8b}$$

and assume that $g(y^*,z^*)=0$. If the matrix of partial derivatives $g_z(y^*,z^*)$ is invertible, then (8) is said to have differentiation index one around (y^*,z^*). This is supported on the fact that one differentiation in (8b) suffices to obtain an explicit underlying ODE

$$\begin{aligned} y' &= f(y,z) \\ z' &= -g_z^{-1}(y,z)g_y(y,z)f(y,z), \end{aligned}$$

for which $g=0$ is an invariant comprising the solutions of (8).

However, the differentiation index notion becomes more intricate in higher index contexts (except if the system is in Hessenberg form) or when the DAE is not semiexplicit. In a general setting, the differentiation index is defined by means of a *derivative array* (see [5, 8, 9]) and its computation becomes more difficult. The derivative array approach is also known to impose more smoothness requirements than necessary; this turns out to be a problem in applications in which the functions involved exhibit low smoothness properties.

The tractability index, supported on a projector-based framework [18, 19, 29, 30, 31, 33], overcomes some of these limitations. The projector-based approach provides an index characterization in terms of the original problem unknowns, leading to a precise functional description of the solution behavior under mild smoothness requirements. The solvability properties of the DAE can be unveiled by means of a *decoupling* of the different solution components (cf. [31, 33, 42]) without recourse to coordinate changes, which are difficult to perform in practice. The tractability index has been proved to be a very useful tool in circuit theory [15, 32, 47, 48]; see also [14, 20, 21, 40, 43]. Our attention will be focused on this index notion, which is discussed in detail below.

3.2 The tractability index

We compile below the key ideas supporting the tractability index notion, together with some technical results which will be useful in the index analysis of Section 4. Note that the index of DAE models arising from a very broad class of strictly passive circuits is known to be no greater than two, and for this reason we restrict the introduction of the tractability index to problems with index ≤ 2, avoiding some technical difficulties which arise in the general case. The importance of the matrices $E_i(x)$ introduced below emanates from the fact that their invertibility (which defines the tractability index of the DAE) supports a decoupling procedure which unveils the solutions of the system; cf. [29, 30, 33, 42, 48].

Assume that $C(v_c)$ and $L(i_l)$ are non-singular; this condition will be met in our working setting because of the assumption that these matrices are positive definite. It follows that the kernel of the leading matrix $E(x)$ is constant. Write $F(x) = -f'(x)$ and let Q be a constant projector onto $\ker E(x)$ (check the form of these matrices in equations (12) and (13) within subsection 4.1 below). The quasilinear DAE (7) is said to have tractability index one if

$$E_1(x) = E(x) + F(x)Q$$

is non-singular.

Suppose now that $E_1(x)$ has constant (non-maximal) rank and that there exists a continuous projector $Q_1(x)$ onto $\ker E_1(x)$. Write $F_1(x) = F(x)P$, with $P = I - Q$. System (7) is then said to have tractability index two if

$$E_2(x) = E_1(x) + F_1(x)Q_1(x)$$

is non-singular. Note that this definition of the index is simpler than the one for general nonlinear DAEs, being feasible because of the special form of the circuit equations; cf. [48, Remark A.18].

These notions mean that, in this context, the computation of the tractability index relies on the construction of the matrices E_1 and (when E_1 is singular) E_2, by means of the projectors Q, Q_1 onto the kernel of the leading matrices E, E_1. From a local point of view this is equivalent to the computation of the Kronecker index [17] of the matrix pencil $\{E(x^*),\ F(x^*)\}$ at any $x = x^*$, as stated in the following result [19, Theorem 3].

Lemma 3. *Let $\{E,\ F\}$ be a matrix pencil with singular E, and assume that Q is any projector onto* $\ker E$*. Then the pencil is regular with Kronecker index one if and only if*

$$E_1 = E + FQ \tag{9}$$

is a non-singular matrix.

If E_1 is singular, denote $P = I - Q$ and let Q_1 be any projector onto $\ker E_1$*. Set $F_1 = FP$. Then the pencil is regular with Kronecker index two if and only if*

$$E_2 = E_1 + F_1Q_1 \tag{10}$$

is non-singular.

These properties do not depend on the specific choices of the projectors Q and Q_1.

Previous characterizations of the tractability index in circuit theory rely on this construction; cf. [15, 43, 47, 48]. However, the following result of Griepentrog and März [19, Theorem 9] will turn out to be useful in our analysis.

Lemma 4. *Let $\{E,\ F\}$ be a matrix pencil with singular E, and assume that R is any projector along* $\operatorname{im} E$*. Then the pencil is regular with Kronecker index one if and only if*

$$E_1^{[R]} = E + RF$$

is a non-singular matrix.

If $E_1^{[R]}$ is singular, let R_1 be any projector along $\operatorname{im} E_1^{[R]}$*. Then the pencil is regular with Kronecker index two if and only if*

$$E_2^{[R]} = E_1^{[R]} + R_1(I - R)F$$

is non-singular.

These properties do not depend on the specific choices of the projectors R and R_1.

Actually, it will be helpful to combine both constructions, as stated in the following remark which follows immediately from the results of [19].

Remark 1. *Provided that E_1 in (9) is a singular matrix, and letting R_1 be a projector along* $\operatorname{im} E_1$*, then the pencil $\{E,\ F\}$ has Kronecker index two if and only if*

$$\tilde{E}_2 = E_1 + R_1F_1$$

is non-singular. Here F_1 stands for $FP = F(I - Q)$, as in (10).

In index analyses it is often useful to use the invariance of the tractability index under pre- and post-multiplication by non-singular matrices. This is a well-known result both for the Kronecker and the tractability indices, but Lemmas 3 and 4 allow for a straightforward proof in the tractability index framework, which is included here for the sake of completeness. Notice, again, that the result is true for arbitrary index even though we restrict the attention to index one and index two cases.

Lemma 5. *If a matrix pencil $\{E,\ F\}$ is regular with Kronecker index one or two, then so it is the pencil $\{\hat{E},\ \hat{F}\}$ with $\hat{E} = GAH$, $\hat{F} = GFH$, G and H being non-singular matrices.*

Proof. Premultiplication by G does not alter the tractability index in the light of Lemma 3. Indeed, set $\bar{E} = GE$, $\bar{F} = GF$. The fact that $\ker E = \ker \bar{E}$ implies that a projector Q onto $\ker E$ is also a projector onto $\ker \bar{E}$, and hence

$$\bar{E}_1 = \bar{E} + \bar{F}Q = GE + GFQ = GE_1.$$

It follows that E_1 is non-singular if and only if so it is $\bar{E}_1$. If they are not, a projector Q_1 onto $\ker E_1$ also projects onto $\ker \bar{E}_1$, yielding

$$\bar{E}_2 = \bar{E}_1 + \bar{F}_1 Q_1 = GE_1 + GF_1 Q_1 = GE_2,$$

which completes the proof concerning premultiplication by G.

Postmultiplication by H does not change the index, either. This follows easily from Lemma 4, as detailed in the sequel. Writing $\hat{E} = \bar{E}H$, $\hat{F} = \bar{F}H$, we have $\operatorname{im} \hat{E} = \operatorname{im} \bar{E}$ and so a projector R along $\operatorname{im} \bar{E}$ also projects along $\operatorname{im} \hat{E}$. This yields

$$\hat{E}_1 = \hat{E} + R\hat{F} = \bar{E}H + R\bar{F}H = \bar{E}_1 H.$$

Therefore $\hat{E}_1$ is non-singular if and only if so it is $\bar{E}_1$. Analogously, a projector R_1 along $\operatorname{im} \bar{E}_1$ will also be a projector along $\operatorname{im} \hat{E}_1$, and then

$$\hat{E}_2 = \hat{E}_1 + R_1 \hat{F}_1 = \bar{E}_1 H + R_1 \bar{F}_1 H = \bar{E}_2 H,$$

an identity which completes the proof. □

4 Index characterization

The material presented in Sections 2 and 3 provides the basis for the characterization of the tractability index of several nodal models of strictly passive circuits including memristors. In subsection 4.1 we prove that under a charge-control assumption on memristors the index of system (6) is two, regardless of the circuit topology, as far as voltage source loops and current source cutsets are excluded. Still for charge-controlled memristors, the elimination of fluxes makes the model index one in the absence of VC-loops and IL-cutsets, and index two in the presence of any of these configurations. This is detailed in subsection 4.2. Finally, in subsection 4.3 we show that analogous results hold for flux-controlled memristors.

The following hypotheses will be assumed throughout, without further explicit mention. The positive definiteness of the circuit matrices expresses a strict passivity assumption, whereas the absence of V-loops and I-cutsets is a standard consistency requirement. For notational simplicity we remove the variables on which the circuit matrices depend.

Working hypotheses. *The conductance, memristance, capacitance and inductance matrices G, M, C, L will be assumed to be positive definite, and the circuits will have neither voltage sources loops nor current sources cutsets.*

4.1 The charge-flux model

Under a charge-control assumption on memristors, the model (6) takes the form

$$\begin{aligned}
C(v_c)v_c' &= i_c && \text{(11a)}\\
L(i_l)i_l' &= A_l^T e && \text{(11b)}\\
\varphi_m' &= A_m^T e && \text{(11c)}\\
q_m' &= i_m && \text{(11d)}\\
0 &= A_r\gamma(A_r^T e) + A_c i_c + A_l i_l + A_m i_m + A_v i_v + A_i i_s(t) && \text{(11e)}\\
0 &= v_c - A_c^T e && \text{(11f)}\\
0 &= v_s(t) - A_v^T e && \text{(11g)}\\
0 &= \varphi_m - \phi(q_m). && \text{(11h)}
\end{aligned}$$

The flux-controlled case will be discussed in subsection 4.3.

Theorem 1. *Under the working hypotheses stated above, the nodal system (11) is index two.*

Proof. In the light of the quasilinear form (cf. (7)) of system (11), write

$$E = \begin{pmatrix} C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_m^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \\ 0 & A_l & 0 & 0 & A_r G A_r^T & A_c & A_v & A_m \\ I & 0 & 0 & 0 & -A_c^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & I & -M & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

Here F stands for the derivative $-f'(x)$, where f comes from the right-hand side of (11); we have freely changed the sign in some of the equations for simplicity. Columns are arranged according to the order of variables v_c, i_l, φ_m, q_m, e, i_c, i_v, i_m.

Since C and L are positive definite and hence non-singular, a projector Q onto $\ker E$ is

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}, \tag{13}$$

which leads, via the relations $E_1 = E + FQ$, $F_1 = FP = F(I - Q)$, to

$$E_1 = \begin{pmatrix} C & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & L & 0 & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -A_m^T & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & A_rGA_r^T & A_c & A_v & A_m \\ 0 & 0 & 0 & 0 & -A_c^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_l & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & -M & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is not a simple matter to compute a projector Q_1 onto $\ker E_1$. We may, however, turn our attention to Remark 1 in order to use instead a projector R_1 along the image of E_1, which in this case is easier to compute. The key aspect in this regard is that a vector $(x_c, x_l, x_m, y_m, y_e, y_c, y_v, z_m)$ belonging to the left-kernel of E_1 is defined by the equations $x_c = x_l = x_m = y_m = 0$, together with

$$y_e^T A_r G A_r^T - y_c^T A_c^T - y_v^T A_v^T = 0 \tag{14a}$$
$$y_e^T A_c = 0 \tag{14b}$$
$$y_e^T A_v = 0 \tag{14c}$$
$$y_e^T A_m = 0. \tag{14d}$$

Multiplying (14a) by y_e and using (14b), (14c) we get $y_e^T A_r G A_r^T y_e = 0$ and, from the positive definiteness of G, $y_e^T A_r = 0$. Together with equations (14b), (14c) and (14d), this indicates that nontrivial values for y_e are linked to the existence of IL-cutsets, according to Lemma 1. Additionally, the relation $y_e^T A_r = 0$ transforms the transposed form of (14a) into $A_c y_c + A_v y_v = 0$, showing that non-vanishing values of (y_c, y_v) arise from the presence of VC-loops (cf. Lemma 2). Note additionally that (14) imposes no restriction on z_m.

These remarks will pave the way for the construction of a projector R_1 along $\operatorname{im} E_1$. Let $\hat{Q}$ be a projector onto $\ker (A_c \ A_v)$, and split it in the form

$$\hat{Q} = \begin{pmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{21} & \hat{Q}_{22} \end{pmatrix}. \tag{15}$$

For later use, notice that

$$A_c\hat{Q}_{11} + A_v\hat{Q}_{21} = 0 \qquad (16a)$$

$$A_c\hat{Q}_{12} + A_v\hat{Q}_{22} = 0. \qquad (16b)$$

Let in turn $\bar{Q}$ be a projector onto $\ker(A_r\ A_c\ A_v\ A_m)^T$, and also for later use keep in mind that

$$A_r^T\bar{Q} = A_c^T\bar{Q} = A_v^T\bar{Q} = A_m^T\bar{Q} = 0. \qquad (17)$$

By construction, $\hat{Q}^T$ and $\bar{Q}^T$ are projectors along $\text{im}\,(A_c\ A_v)^T$ and $\text{im}\,(A_r\ A_c\ A_v\ A_m)$, respectively. A projector along $\text{im}\,E_1$ is then

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{Q}_{11}^T & \hat{Q}_{21}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{Q}_{12}^T & \hat{Q}_{22}^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}, \qquad (18)$$

so that

$$\tilde{E}_2 = E_1 + R_1F_1 = \begin{pmatrix} C & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & L & 0 & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & I & 0 & -A_m^T & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & -I \\ 0 & \bar{Q}^TA_l & 0 & 0 & A_rGA_r^T & A_c & A_v & A_m \\ \hat{Q}_{11}^T & 0 & 0 & 0 & -A_c^T & 0 & 0 & 0 \\ \hat{Q}_{12}^T & 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & I & -M & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix $\tilde{E}_2$ is non-singular if and only if so it is the Schur complement [24] of the upper-left 4×4 block; this complement can be easily checked to read

$$\begin{pmatrix} A_rGA_r^T + \bar{Q}^TA_lL^{-1}A_l^T & A_c & A_v & A_m \\ -A_c^T & \hat{Q}_{11}^TC^{-1} & 0 & 0 \\ -A_v^T & \hat{Q}_{12}^TC^{-1} & 0 & 0 \\ A_m^T & 0 & 0 & -M \end{pmatrix}. \qquad (19)$$

This matrix will be non-singular if and only if its left-kernel amounts to the null vector. In order to prove the index two nature of the system we then need to show that the unique solution to the system

$$x^T[A_rGA_r^T + \bar{Q}^TA_lL^{-1}A_l^T] - y^TA_c^T - z^TA_v^T + u^TA_m^T = 0 \qquad (20a)$$

$$x^TA_c + y^T\hat{Q}_{11}^TC^{-1} + z^T\hat{Q}_{12}^TC^{-1} = 0 \qquad (20b)$$

$$x^TA_v = 0 \qquad (20c)$$

$$x^TA_m - u^TM = 0 \qquad (20d)$$

is $x = y = z = u = 0$

In the light of (20d), equation (20a) can be rewritten as

$$x^T[A_rGA_r^T + A_mWA_m^T + \bar{Q}^TA_lL^{-1}A_l^T] - y^TA_c^T - z^TA_v^T \ = \ 0, \tag{21}$$

where we have made use of the memductance $W = M^{-1}$. Let us then multiply (21) by $\bar{Q}x$ and, using (17), derive $x^T\bar{Q}^TA_lL^{-1}A_l^T\bar{Q}x = 0$. The fact that L^{-1} is positive definite implies $x^T\bar{Q}^TA_l = 0$, so that (21) amounts to

$$x^T[A_rGA_r^T + A_mWA_m^T] - y^TA_c^T - z^TA_v^T \ = \ 0. \tag{22}$$

In turn, multiplying (20b) by CA_c^Tx, and using (16) together with $A_v^Tx = 0$ from (20c), we derive the relation $x^TA_cCA_c^Tx = 0$. This implies

$$x^TA_c = 0, \tag{23}$$

because C is positive definite. Multiplying (22) by x we then derive, using (20c) and (23),

$$x^T[A_rGA_r^T + A_mWA_m^T]x \ = \ 0. \tag{24}$$

Because of the assumption that both the conductance and the memductance matrices are positive definite, (24) yields

$$x^TA_r \ = \ 0 \tag{25a}$$

$$x^TA_m \ = \ 0. \tag{25b}$$

Equations (20c), (23), (25a) and (25b) show that $x \in \ker(A_r\ A_c\ A_v\ A_m)^T$, so that $\bar{Q}x = x$, i.e., $x^T\bar{Q}^T = x^T$. The identity $x^T\bar{Q}^TA_l = 0$ derived above can be then simplified to $x^TA_l = 0$, meaning that x actually belongs to $\ker(A_r\ A_c\ A_v\ A_m\ A_l)^T$. Since cutsets defined only by current sources are precluded, we derive $x = 0$ from Lemma 1. In turn, (20d) yields $u = 0$.

It just remains to show that also y and z vanish. Equations (25a) and (25b) make it possible to recast (22) as $-y^TA_c^T - z^TA_v^T = 0$, that is, $A_cy + A_vz = 0$. The latter means that $y = \hat{Q}_{11}y + \hat{Q}_{12}z$ because $(y, z) \in \ker(A_c\ A_v)$ and $\hat{Q}$ is a projector onto this space. Now, using (23) we get from (20b) the relation $y^T\hat{Q}_{11}^T + z^T\hat{Q}_{12}^T = 0$, and then $y = \hat{Q}_{11}y + \hat{Q}_{12}z = 0$. This means in turn that $A_vz = 0$, a condition which implies $z = 0$ because of the absence of voltage source loops and Lemma 2. This completes the proof. □

4.2 The charge-oriented model

Eliminating the flux from system (11) we arrive at a model which is index one in the absence of VC-loops and IL-cutsets, as detailed below. A proof based just on the Q-projectors onto the kernel of the leading matrices is now feasible.

Using the memristor characteristic (11h), we may replace (11c) by $M(q_m)q_m' = A_m^Te$ and, using (11d), recast this relation as $M(q_m)i_m - A_m^Te = 0$. This yields the *charge-oriented* *system*

$$\begin{aligned}
C(v_c)v_c' &= i_c && (26a)\\
L(i_l)i_l' &= A_l^T e && (26b)\\
q_m' &= i_m && (26c)\\
0 &= A_r\gamma(A_r^T e) + A_c i_c + A_l i_l + A_m i_m + A_v i_v + A_i i_s(t) && (26d)\\
0 &= v_c - A_c^T e && (26e)\\
0 &= v_s(t) - A_v^T e && (26f)\\
0 &= M(q_m)i_m - A_m^T e. && (26g)
\end{aligned}$$

Theorem 2. *System (26) is index one in the absence of VC-loops and IL-cutsets, and index two in the presence of at least one VC-loop including capacitors and/or at least one IL-cutset including inductors.*

Proof. Now we have

$$E = \begin{pmatrix} C & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & L & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 & -I & 0 & 0\\ 0 & 0 & 0 & -A_l^T & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -I\\ 0 & A_l & 0 & A_rGA_r^T & A_c & A_v & A_m\\ I & 0 & 0 & -A_c^T & 0 & 0 & 0\\ 0 & 0 & 0 & -A_v^T & 0 & 0 & 0\\ 0 & 0 & K & -A_m^T & 0 & 0 & M \end{pmatrix}, \qquad (27)$$

with $K = \partial(M(q_m)i_m)/\partial q_m$. The order of variables is v_c, i_l, q_m, e, i_c, i_v, i_m. Choosing the projector

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}, \qquad (28)$$

we now get for $E_1 = E + FQ$, $F_1 = FP = F(I-Q)$ the expressions

$$E_1 = \begin{pmatrix} C & 0 & 0 & 0 & -I & 0 & 0\\ 0 & L & 0 & -A_l^T & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 & -I\\ 0 & 0 & 0 & A_rGA_r^T & A_c & A_v & A_m\\ 0 & 0 & 0 & -A_c^T & 0 & 0 & 0\\ 0 & 0 & 0 & -A_v^T & 0 & 0 & 0\\ 0 & 0 & 0 & -A_m^T & 0 & 0 & M \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & A_l & 0 & 0 & 0 & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & K & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix E_1 will be non-singular if and only if so it is

$$\begin{pmatrix} A_r G A_r^T & A_c & A_v & A_m \\ -A_c^T & 0 & 0 & 0 \\ -A_v^T & 0 & 0 & 0 \\ -A_m^T & 0 & 0 & M \end{pmatrix}.$$

A vector (x, y, z, u) of the kernel of this matrix must verify

$$A_r G A_r^T x + A_c y + A_v z + A_m u = 0 \quad (29a)$$
$$-A_c^T x = 0 \quad (29b)$$
$$-A_v^T x = 0 \quad (29c)$$
$$-A_m^T x + M u = 0. \quad (29d)$$

Using (29d), (29a) reads

$$(A_r G A_r^T + A_m W A_m^T)x + A_c y + A_v z = 0, \quad (30)$$

written in terms of the memductance $W = M^{-1}$. This equation premultiplied by x^T yields, using (29b) and (29c), the relation $x^T(A_r G A_r^T + A_m W A_m^T)x = 0$ which, since both G and W are positive definite, implies

$$A_r^T x = 0 \quad (31a)$$
$$A_m^T x = 0. \quad (31b)$$

In turn these relations simplify (30) to

$$A_c y + A_v z = 0. \quad (32)$$

In the light of equations (29b), (29c), (31a) and (31b), a non-vanishing x necessarily describes the existence of an IL-cutset (cf. Lemma 1), whereas (32) shows that non-null vectors (y, z) correspond to VC-loops, according to Lemma 2. In the absence of both IL-cutsets and VC-loops, the matrix E_1 is non-singular and the system is therefore index one.

In the presence of IL-cutsets and/or VC-loops, the remarks above make it easy to construct a projector Q_1 onto $\ker E_1$. Let us use again a projector $\hat{Q}$ onto $\ker (A_c \ A_v)$, keeping in mind the splitting (15), and also a projector $\bar{Q}$ onto $\ker (A_r \ A_c \ A_v \ A_m)^T$. A projector Q_1 onto $\ker E_1$ is

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & C^{-1}\hat{Q}_{11} & C^{-1}\hat{Q}_{12} & 0 \\ 0 & 0 & 0 & L^{-1}A_l^T\bar{Q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{Q} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{Q}_{11} & \hat{Q}_{12} & 0 \\ 0 & 0 & 0 & 0 & \hat{Q}_{21} & \hat{Q}_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (33)$$

and then $E_2 = E_1 + F_1 Q_1 = E_1 + F(I-Q)Q_1$ may be checked to read

$$E_2 = \begin{pmatrix} C & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & L & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & A_r G A_r^T + A_l L^{-1} A_l^T \bar{Q} & A_c & A_v & A_m \\ 0 & 0 & 0 & -A_c^T & C^{-1}\hat{Q}_{11} & C^{-1}\hat{Q}_{12} & 0 \\ 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_m^T & 0 & 0 & M \end{pmatrix},$$

which is non-singular if and only if so it is the lower-right 4×4 block. However, disregarding minus signs, the transpose of this matrix has the form depicted in (19), and therefore it can be proved to be non-singular exactly as in subsection 4.1. This shows that in the presence of IL-cutsets and/or VC-loops, system (26) is index two. □

Theorem 2 indicates that, in topologically nondegenerate circuits (that is, circuits without VC-loops and IL-cutsets), the dynamical degree of freedom equals the number of capacitors, inductors and memristors, since a state space reduction in terms of v_c, i_l and q_m is (at least theoretically) possible in the index one setting. It can be shown that in the index two cases associated with topologically degenerate configurations, the dynamical degree of freedom is defined by the number of capacitors in a normal tree (namely, a tree with the maximum possible number of capacitors and the minimum possible number of inductors) and inductors in a normal cotree plus the total number of memristors.

4.3 Flux-controlled memristors

The results above also apply to flux-controlled memristors, as detailed in the sequel. Theorems 3 and 4 below follow easily from the corresponding charge-controlled results stated in Theorems 1 and 2, respectively.

A flux-control assumption on memristors yields the system

$$\begin{aligned} C(v_c)v_c' &= i_c & \text{(34a)} \\ L(i_l)i_l' &= A_l^T e & \text{(34b)} \\ \varphi_m' &= A_m^T e & \text{(34c)} \\ q_m' &= i_m & \text{(34d)} \\ 0 &= A_r \gamma(A_r^T e) + A_c i_c + A_l i_l + A_m i_m + A_v i_v + A_i i_s(t) & \text{(34e)} \\ 0 &= v_c - A_c^T e & \text{(34f)} \\ 0 &= v_s(t) - A_v^T e & \text{(34g)} \\ 0 &= q_m - \sigma(\varphi_m). & \text{(34h)} \end{aligned}$$

Theorem 3. *The flux-controlled system (34) is index two.*

Proof. Write

$$E = \begin{pmatrix} C & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_m^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \\ 0 & A_l & 0 & 0 & A_rGA_r^T & A_c & A_v & A_m \\ I & 0 & 0 & 0 & -A_c^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & -W & I & 0 & 0 & 0 & 0 \end{pmatrix},$$

again arranging columns according to the order v_c, i_l, φ_m, q_m, e, i_c, i_v, i_m. The fact that the system is still index two is an immediate consequence of Lemma 5, since the matrices E and F are obtained from the ones corresponding to the charge-controlled model (cf. (12)) just by premultiplying these by the non-singular matrix block-diag$(I, -W)$, where $W = M^{-1}$ is the memductance matrix. □

Finally, using (34d) and (34h) we may eliminate the memristors' charges from the model. This yields the *flux-oriented system*

$$C(v_c)v_c' = i_c \tag{35a}$$
$$L(i_l)i_l' = A_l^T e \tag{35b}$$
$$\varphi_m' = A_m^T e \tag{35c}$$
$$0 = A_r\gamma(A_r^T e) + A_c i_c + A_l i_l + A_m i_m + A_v i_v + A_i i_s(t) \tag{35d}$$
$$0 = v_c - A_c^T e \tag{35e}$$
$$0 = v_s(t) - A_v^T e \tag{35f}$$
$$0 = i_m - W(\varphi_m)A_m^T e. \tag{35g}$$

Theorem 4. *System (35) is index one in the absence of VC-loops and IL-cutsets, and index two in the presence of at least one VC-loop including capacitors and/or at least one IL-cutset including inductors.*

Proof. The matrices E and F read

$$E = \begin{pmatrix} C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_m^T & 0 & 0 & 0 \\ 0 & A_l & 0 & A_rGA_r^T & A_c & A_v & A_m \\ I & 0 & 0 & -A_c^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & \tilde{K} & -WA_m^T & 0 & 0 & I \end{pmatrix},$$

with $\tilde{K} = -\partial(W(\varphi_m)A_m^T e)/\partial\varphi_m$. We have arranged the columns according to the order v_c, i_l, φ_m, e, i_c, i_v, i_m. The matrices E_1, F_1 read in this case, using the projector Q defined in

(28),

$$E_1 = \begin{pmatrix} C & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & L & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & I & -A_m^T & 0 & 0 & 0 \\ 0 & 0 & 0 & A_rGA_r^T & A_c & A_v & A_m \\ 0 & 0 & 0 & -A_c^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -WA_m^T & 0 & 0 & I \end{pmatrix}, \quad F_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_l & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{K} & 0 & 0 & 0 & 0 \end{pmatrix},$$

whereas Q_1 from (33) yields

$$E_2 = \begin{pmatrix} C & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & L & 0 & -A_l^T & 0 & 0 & 0 \\ 0 & 0 & I & -A_m^T & 0 & 0 & 0 \\ 0 & 0 & 0 & A_rGA_r^T + A_lL^{-1}A_l^T\bar{Q} & A_c & A_v & A_m \\ 0 & 0 & 0 & -A_c^T & C^{-1}\hat{Q}_{11} & C^{-1}\hat{Q}_{12} & 0 \\ 0 & 0 & 0 & -A_v^T & 0 & 0 & 0 \\ 0 & 0 & 0 & -WA_m^T & 0 & 0 & I \end{pmatrix}.$$

The non-singularity of E_1 and E_2 relies on the lower-right 4×4 blocks, which coincide with the ones arising in the charge-oriented case except for the fact that the last rows are multiplied by the non-singular matrix W. Therefore, the index behavior is the same as in the charge-oriented case and the proof is complete. □

5 Example

Consider the circuit depicted in Figure 1.

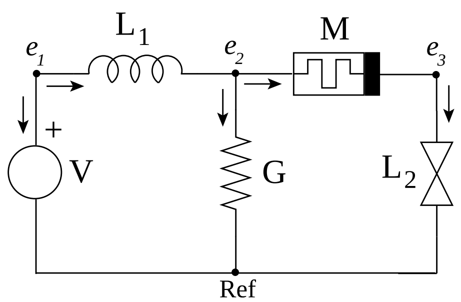

Figure 1: A memristive circuit including a Josephson junction.

The circuit includes a voltage source V, a linear inductor with inductance L_1, a linear resistor with conductance G, a memristor labelled as M and a Josephson junction labelled as L_2. The node potentials are e_1, e_2, e_3, and the arrows indicate the reference direction in each circuit branch. We will let i_v, i_1, i_m and i_2 stand for the current through the voltage source, the inductor, the memristor and the Josephson junction, respectively.

Following [45], the memristance is assumed to have the charge-controlled form $M(q_m) = k_1 - k_2 q_m$, where k_1 and k_2 are physical constants. The device design restricts the memristance $M(q_m)$ to take positive values. In turn, the Josephson junction consists of two superconductors separated by an oxide barrier [12], and is governed by a current-flux relation of the form $i_2 = I_0 \sin k\varphi$ for certain device parameters I_0, k. For our present purposes it will be enough to consider this device as a nonlinear inductor with a (current-dependent) incremental inductance L_2; we will assume that $L_2 > 0$ in the operating region, although some remarks concerning cases in which L_2 may become negative will be discussed later.

We will use this circuit to illustrate the notions and results discussed in previous sections; specifically, we aim to illustrate the form of the nodal model and the expressions that the main matrices in the index analysis take. For the sake of brevity we restrict the attention to the charge-oriented model of subsection 4.2; our ultimate goal is to show how the index characterization presented in Theorem 2 applies to this problem. The reader should have no difficulty in extending this analysis to the other models discussed in the present paper.

The model (26) reads for this circuit

$$L_1 i_1' = e_1 - e_2 \tag{36a}$$
$$L_2(i_2) i_2' = e_3 \tag{36b}$$
$$q_m' = i_m \tag{36c}$$
$$0 = i_1 + i_v \tag{36d}$$
$$0 = G e_2 + i_m - i_1 \tag{36e}$$
$$0 = -i_m + i_2 \tag{36f}$$
$$0 = v_s(t) - e_1 \tag{36g}$$
$$0 = M(q_m) i_m - e_2 + e_3, \tag{36h}$$

where $v_s(t)$ is the voltage in the source.

Index one

If the variables are arranged according to the order i_1, i_2, q_m, e_1, e_2, e_3, i_v, i_m then the left- and right-hand sides of (36) confer to the matrices E and F the form

$$E = \begin{pmatrix} L_1 & & & & & & & \\ & L_2 & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & G & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & M'(q_m) i_m & 0 & -1 & 1 & 0 & M(q_m) \end{pmatrix}.$$

Using a projector Q onto $\ker E$ of the form depicted in (28), that means

$$Q = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{pmatrix},$$

it is easy to check that the matrix $E_1 = E + FQ$ reads

$$E_1 = \begin{pmatrix} L_1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & G & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & M \end{pmatrix}. \tag{37}$$

Elementary computations show that the determinant of (37) equals the product $L_1 L_2 G$.

Assume that all devices are (locally) strictly passive. This means that the conductance G, the memristance M and the inductances L_1, L_2 are positive, making the determinant of (37) non-null. In this setting the model (36) is therefore index one. This is consistent with the absence of VC-loops and IL-cutsets, according to Theorem 2. In this situation the dynamical degree of freedom of the circuit is three, the dynamics being defined by the currents in the inductor and in the Josephson junction and the charge in the memristor.

Index two

Assume now that the resistor is open-circuited, so that $G = 0$. This makes E_1 in (37) a singular matrix. A projector onto the kernel of this matrix, having the form depicted in (33), reads

$$Q_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & -L_1^{-1} & -L_1^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & L_2^{-1} & L_2^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{38}$$

and yields for the matrix $E_2 = E_1 + F_1 Q_1 = E_1 + F(I - Q)Q_1$ the expression

$$E_2 = \begin{pmatrix} L_1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -(2L_1)^{-1} & -(2L_1)^{-1} & 1 & 0 \\ 0 & 0 & 0 & 0 & (2L_1)^{-1} & (2L_1)^{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 & (2L_2)^{-1} & (2L_2)^{-1} & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & M \end{pmatrix}.$$

The determinant of E_2 is $L_1 L_2 (L_1^{-1} + L_2^{-1})$. Provided that all inductances are positive, then so it is this determinant, making the model (36) index two. Again, this result can be easily derived from the characterization presented in Theorem 2, since the removal of G in Figure 1 results in an L-cutset defined by the linear inductor and the Josephson junction. In this index two configuration the currents through the inductor and the Josephson junction become redundant and the dynamical degree of freedom is reduced to two.

As a final remark, note that in this case we can not relax the assumption of positiveness on the inductances to just requiring them not to vanish. Actually, in operating regions in which the Josephson junction is locally active, its incremental inductance L_2 becomes negative; it may reach the value $-L_1$ and, in this situation, the determinant of E_2 does vanish, making the index two condition fail. Different techniques, beyond the scope of this paper, are necessary in order to obtain an index characterization of these models when the strict passivity assumptions are not met.

6 Concluding remarks

Memristors are likely to play a relevant role in electrical engineering and in electronics in the near future. We have shown in this paper how to accommodate these devices in the semistate (or differential-algebraic) models of electrical circuits arising in nodal analysis. A key role in the characterization of different features of these models is played by their *index*. We have proved that the models which combine memristors' fluxes and charges are index two. Eliminating the fluxes of charge-controlled memristors (and viceversa) we get index one systems in topologically nondegenerate circuits, and index two models otherwise.

Our results should be of interest in the numerical simulation of circuits with memristors, and also in future studies of analytical aspects of these circuits, involving e.g. oscillations, stability issues or bifurcation phenomena. Some open lines for future research concern the characterization of the index of memristive circuits without strict passivity assumptions on the circuit matrices, as well as the analysis of other model families and of distributed memristive systems.

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