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TRANSCRITICAL BIFURCATION WITHOUT PARAMETERS IN MEMRISTIVE CIRCUITS*

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Abstract. The transcritical bifurcation without parameters (TBWP) describes a stability change along a line of equilibria, resulting from the loss of normal hyperbolicity at a given point of such a line. Memristive circuits systematically yield manifolds of nonisolated equilibria, and in this paper we address a systematic characterization of the TBWP in circuits with a single memristor. To achieve this we develop two mathematical results of independent interest; the first is an extension of the TBWP theorem to explicit ordinary differential equations (ODEs) in arbitrary dimension; the second result drives the characterization of this phenomenon to semiexplicit differential-algebraic equations (DAEs), which provide the appropriate framework for the analysis of circuit dynamics. In the circuit context the analysis is performed in graph-theoretic terms: in this setting, our first working scenario is restricted to passive problems (an exception is made for the bifurcating memristor), and in a second step some results are presented for the analysis of nonpassive cases. The latter context is illustrated by means of a memristive neural network model.

Key words. manifold of equilibria, normal hyperbolicity, transcritical bifurcation without parameters, differential-algebraic equation, nonlinear circuit, memristor

AMS subject classifications. 34A09, 34C45, 34D35, 37G10, 94C05, 94C15

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1. Introduction.

Memristors and other related electronic devices [11, 17, 53] are known to systematically exhibit manifolds of nonisolated equilibrium points. This is a consequence of the form of the voltage-current characteristic of memristive devices. As shown in [45], in the absence of certain configurations equilibrium manifolds of strictly locally passive memristive circuits are normally hyperbolic; that is, all remaining eigenvalues of the linearized vector field (except for those whose eigenvectors span the tangent space to the equilibrium manifold) are away from the imaginary axis. From a qualitative point of view it is therefore natural to examine what happens in the memristive circuit dynamics when the aforementioned passivity assumption does not hold.

This problem must be framed in the theory of *bifurcation without parameters* originally introduced in the seminal papers [19, 20, 21]; cf. also the recent book [31]. When normal hyperbolicity fails, a change in the local qualitative properties typically occurs along the equilibrium manifold—hence the "bifurcation without parameters" term. In this context, the most basic phenomenon is the transcritical bifurcation without parameters (TBWP), which describes the transition of one eigenvalue through the origin under certain local conditions on the vector field. Our purpose in this paper is to present a systematic circuit-theoretic characterization of this bifurcation for memristive circuits. Due to the pervasive presence of nonisolated equilibria in memristive circuits, this is the most elementary phenomenon responsible for a stability

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loss in nonlinear circuits with memristors, and its analysis seems therefore to be very relevant for the development of the qualitative theory of memristive circuits.

This is actually a threefold goal. First, the characterization of the TBWP in [19] (and also in [31]) is only addressed for two-dimensional dynamics. However, most nonlinear circuits involve a large number of dynamic variables, and a two-dimensional model reduction is rarely feasible. For their results to apply to higher dimensional problems, the authors assume in [19, 31] that a prior reduction to a center two-dimensional manifold has been performed, but no explicit conditions paving the way for an appropriate reduction are given in arbitrary (finite) dimension. As a somewhat natural (yet not trivial) extension of their characterization we will present a TBWP theorem for explicit ODEs in $\mathbb{R}^n$, addressing the geometrical conditions which allow for a center manifold reduction in which the two-dimensional conditions of [19] do hold. This first goal is tackled in section 2 (cf. Theorem 2).

Many nonlinear circuits do not admit in practice a dynamical description in terms of an explicit ODE. This is clearly the case in large scale integration circuits, for which a state-space description in terms of an explicit ODE model is hardly automatable. For this reason, semistate models based on differential-algebraic equations (DAEs) are often preferred instead [18, 23, 26, 30, 37, 42, 54, 56]. It is therefore of interest to reformulate in DAE terms different analytical results involving dynamical systems if they are intended to apply to general nonlinear circuit models. Many qualitative investigations about nonlinear circuits require a prior reduction to an ODE model (see [12] as a relevant sample in this direction), sometimes involving unnecessarily restrictive hypotheses which, in addition, could make the analysis more difficult. In the DAE framework the approach is different: instead of trying to drive a model to the ODE context in order to apply a given known result, it is more convenient to extend such a result to the DAE setting, allowing for a direct application to a semistate model. In this direction, our second goal is to drive the TBWP theorem to the semiexplicit DAE setting, a task which is accomplished in section 3 and, specifically, in Theorem 3.

As indicated above, our third goal is to obtain a characterization of the TBWP for memristive circuits in circuit-theoretic terms. This means that the characterization should be stated in terms of the underlying circuit digraph and the electrical features of the devices. This so-called *structural* approach has its roots in the state-space formulation problem (whose origins can be traced back to [6, 9]) and, more recently, has been successfully applied to the DAE index characterization of circuit models [18, 23, 26, 42, 54, 56, 57]. This approach makes it possible to directly transfer different analytical and qualitative results to circuit simulation programs. Allowed by the TBWP theorem for DAEs obtained in section 3, such a characterization is detailed in section 4 for circuits displaying a line of equilibria (that is, including exactly one memristor), under the assumption that the failing of a passivity assumption on this memristor is the one responsible for the loss of normal hyperbolicity; cf. Theorem 4. The analysis in this section extends some preliminary results presented in [22].

Section 5 discusses this phenomenon relaxing the passivity assumption on the remaining circuit devices and includes an example coming from the theory of memristive neural networks. In section 6 we discuss some relations with a recently introduced flux-charge formalism for the description of circuit dynamics. Finally, section 7 compiles some concluding remarks.

2. The TBWP theorem for explicit ODEs.

2.1. Two-dimensional dynamics. We begin by recalling the characterization of the TBWP in two-dimensional problems presented by Fiedler, Liebscher, and Alexander in [19].

THEOREM 1 (Fiedler, Liebscher, and Alexander, 2000). *Consider the system*

$$x' = \xi_1(x, y), \tag{1a}$$

$$y' = \xi_2(x, y), \tag{1b}$$

with $\xi \in C^2(\mathbb{R}^2, \mathbb{R}^2)$*, and assume that*

1. $\xi(x, 0) = 0$;
2. $\frac{\partial \xi_2}{\partial y}(0, 0) = 0$;
3. $\frac{\partial \xi_1}{\partial y}(0, 0) \neq 0$;
4. $\frac{\partial^2 \xi_2}{\partial x \partial y}(0, 0) \neq 0$.

Then (1) *is locally orbitally* C^1*-equivalent to the normal form* $x' = y$, $y' = xy$ *around the origin.*

Needless to say, it is enough to assume that condition 1 holds for x sufficiently close to 0. Note that $y = 0$ is a line of equilibria for (1) and also for the resulting normal form, and that $\lambda = 0$ is an eigenvalue for the linearization of both systems at any $(x, 0)$. This zero eigenvalue becomes a double (index-two) eigenvalue at $(0, 0)$, in a way such that the second system eigenvalue changes sign along the line of equilibria; specifically, this second eigenvalue is positive (resp., negative) if $x > 0$ (resp., $x < 0$) in the normal form derived in Theorem 1. This means that the line of equilibria is normally hyperbolic for $x \neq 0$, and a stability change occurs along the line of equilibria as a result of the loss of normal hyperbolicity at the origin. This is the transcritical bifurcation without parameters (TBWP).

2.2. The TBWP for explicit ODEs in $\mathbb{R}^n$. Theorem 1 can be extended to explicit ODEs in arbitrary (finite) dimension as follows. Note that a notational abuse is used in different situations throughout the paper, namely writing both $f(x)$ and $f(x_1, \ldots, x_n)$, the latter standing for the (more cumbersome) $f((x_1, \ldots, x_n))$. Obviously, the result below can also be stated for an open set $\Omega \subseteq \mathbb{R}^n$ with $0 \in \Omega$ or for a germ of a map at the origin.

THEOREM 2 (TBWP in $\mathbb{R}^n$). *Assume that* $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ *verifies the following.*

1. $f(x_1, 0, \ldots, 0) = 0$;
2. $f'(0)$ *has a double index-two zero eigenvalue, and* $\operatorname{Re} \lambda \neq 0$ *for the remaining ones;*
3. $f''(0)pq \notin \operatorname{im} f'(0)$ *if* $p \in \ker f'(0) - \{0\}$ *and* $q \in \ker (f'(0))^2 - \ker f'(0)$.

Then there exists a local, two-dimensional, C^2 *center manifold, where the reduced dynamics admits a description in local coordinates of the form* $u' = \xi(u)$*, with* ξ *verifying the conditions of Theorem* 1 *in* $u^* = (0, 0)$.

Proof. The proof relies on the fact that the linear transformation driving the linear part to Jordan form leaves the line of equilibria invariant; note also that condition 3 captures the geometric (transversality) hypothesis which extends to higher-dimensional contexts the second-order condition in Theorem 1. For better clarity we proceed in numbered steps.

1. Let P drive $f'(0)$ to Jordan form: $\tilde{J} = P^{-1} f'(0) P = \left(\begin{smallmatrix} J_0 & 0 \\ 0 & J_h \end{smallmatrix}\right)$, $J_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, J_h hyperbolic.

2. Under the change of coordinates $x = Py$, the system $x' = f(x)$ is transformed into $y' = \tilde{f}(y) = P^{-1}f(Py)$, which reads as

$$u' = \tilde{f}_1(u,v) = J_0 u + \eta(u,v),$$
$$v' = \tilde{f}_2(u,v) = J_h v + \gamma(u,v)$$

with $y = (u,v),\ u \in \mathbb{R}^2,\ v \in \mathbb{R}^{n-2}$, and $\eta'(0,0) = 0$, $\gamma'(0,0) = 0$.

3. From the fact that $\tilde{p} \in \ker \tilde{f}'(0) \Leftrightarrow p = P\tilde{p} \in \ker f'(0)$ it follows that P leaves $\ker f'(0) = \operatorname{span}\{e_1\}$ (with $e_1 = (1,0,\ldots,0)$) invariant, and therefore the equilibrium line of $\tilde{f}$ is $(u_1,0,\ldots,0)$.

4. Using, analogously, the properties $\tilde{q} \in \ker(\tilde{f}'(0))^2 \Leftrightarrow q = P\tilde{q} \in \ker(f'(0))^2$ and $\tilde{w} \in \operatorname{im} \tilde{f}'(0) \Leftrightarrow w = P\tilde{w} \in \operatorname{im} f'(0)$, the condition

$$f''(0)pq \notin \operatorname{im} f'(0) \text{ with } p \in \ker f'(0) - \{0\} \text{ and } q \in \ker(f'(0))^2 - \ker f'(0)$$

yields $\tilde{f}''(0)\tilde{p}\tilde{q} \notin \operatorname{im} \tilde{f}'(0)$, with $\tilde{p} \in \ker \tilde{f}'(0) - \{0\}$ and $\tilde{q} \in \ker(\tilde{f}'(0))^2 - \ker \tilde{f}'(0)$, and, in turn, this leads to

$$\frac{\partial^2 \eta_2}{\partial u_1 \partial u_2}(0,0) \neq 0. \tag{2}$$

5. The system $y' = \tilde{f}(y)$ admits a local center manifold of the form $v = \zeta(u)$, with $\zeta(0) = 0,\ \zeta'(0) = 0$ (see, e.g., [10, 40, 60]). The dynamics on this manifold reads as $u' = \xi(u) = J_0 u + \eta(u, \zeta(u))$.

6. Locally, the curve of equilibria must belong to the center manifold [10, 60], and this yields condition 1 of Theorem 1, that is, $\xi(u_1, 0) = 0$. Additionally, the form of J_0 renders conditions 2–3 trivial, and (2) yields condition 4, that is,

$$\frac{\partial^2 \xi_2}{\partial u_1 \partial u_2}(0,0) \neq 0,$$

since $\eta'(0,0) = 0$, $\zeta'(0,0) = 0$. This completes the proof. □

Remark. Geometrically, condition 3 expresses the transversality at x^* of the center manifold and the so-called singular manifold $\{x \in \mathbb{R}^n \ /\ \det f'(x) = 0\}$ as a consequence of the following well-known property from matrix analysis (we omit the proof; find details, e.g., in [44]).

LEMMA 1. *If* $H \in C^1(\mathbb{R}^m, \mathbb{R}^{n\times n})$, $\operatorname{rk} H(x^*) = n-1$, *and* $p \in \ker H(x^*) - \{0\}$, *then*

$$(H'(x^*)q)p \notin \operatorname{im} H(x^*) \Leftrightarrow (\det H)'(x^*)q \neq 0.$$

Lemma 1 implies in particular that, if $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $\operatorname{rk} f'(x^*) = n-1$, then

$$f''(x^*)pq \notin \operatorname{im} f'(x^*) \Leftrightarrow (\det f')'(x^*)q \neq 0,$$

a condition which, in the setting of Theorem 2, expresses the transversal intersection of the direction spanned by the generalized eigenvector q (hence of the center manifold itself) and the aforementioned singular manifold, as indicated in the remark above. For later use, we also note that for a single-parameter valued matrix map $H \in C^1(\mathbb{R}, \mathbb{R}^{n\times n})$ with $H'(\lambda^*) = n-1$ we have

$$H'(\lambda^*)p \notin \operatorname{im} H(\lambda^*) \Leftrightarrow (\det H)'(\lambda^*) \neq 0.$$

Lemma 1 also shows that condition 3 does not depend on the choice of q, since $\ker f'(0)$ is tangent to the singular manifold, so that $f''(0)pp \in \operatorname{im} f'(0)$, and then, for $\hat{q} = \alpha q + \beta p$ with $\alpha \neq 0$, we have

$$f''(0)p\hat{q} = (\alpha f''(0)pq + \beta f''(0)pp) \notin \operatorname{im} f'(0) \Leftrightarrow f''(0)pq \notin \operatorname{im} f'(0).$$

Note also that the form of $\ker f'(0) = \operatorname{span}\{e_1\}$ makes it possible to simplify the statement of condition 3 to

$$f_{x_1 x}(0)q \notin \operatorname{im} f'(0),$$

where $f_{x_1 x}$ denotes the matrix of partial derivatives $(\frac{\partial^2 f_i}{\partial x_1 \partial x_j})$.

Finally, from the Šošitaĭšvili–Palmer theorem [39], it follows that the normal form for the TBWP in $\mathbb{R}^n$ is

$$x' = y, \tag{3a}$$

$$y' = xy, \tag{3b}$$

$$v' = J_h v. \tag{3c}$$

Certainly, the form of the latter equation may be further simplified to that of a standard node or saddle point, depending on the inertia of J_h.

3. TBWP in semiexplicit DAEs. Along the route indicated in section 1, we extend below the TBWP to the setting of semiexplicit DAEs.

Theorem 3 (TBWP in semiexplicit index-one DAEs). *Let $h \in C^2(\mathbb{R}^{r+p}, \mathbb{R}^r)$, $g \in C^2(\mathbb{R}^{r+p}, \mathbb{R}^p)$, and consider the system*

$$y' = h(y, z), \tag{4a}$$

$$0 = g(y, z). \tag{4b}$$

Write $E = \left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $F = (h, g)$. Assume that $g_z(0,0)$ is nonsingular and that

1. *$h(y_1, 0, 0) = 0$, $g(y_1, 0, 0) = 0$;*
2. *the matrix pencil $\lambda E - F'(0,0)$ has a double index-two zero eigenvalue, and $\operatorname{Re}\lambda \neq 0$ for the remaining eigenvalues;*
3. *$F''(0,0)\overline{pq} \notin \operatorname{im} F'(0,0)$, where*

$$\overline{p} \in \ker F'(0,0) - \{0\}, \ \overline{q} \in \ker (F'(0,0))^2 - \ker F'(0,0). \tag{5}$$

Then, there exists an invariant, two-dimensional, C^2 submanifold of $g(y,z) = 0$ where the dynamics admits a local description of the form $u' = \xi(u)$ with ξ satisfying the conditions of Theorem 1 at the origin.

Before proceeding with the proof we present some auxiliary results.

Lemma 2 (Schur). *Let D be a nonsingular matrix and*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ (M/D) = A - BD^{-1}C, \tag{6}$$

with A (hence M) square. Then $\det M = \det(M/D) \det D$ *and* $\operatorname{cork} M = \operatorname{cork}(M/D)$.

We will make use of this lemma at several points in our analysis, mostly with

$$M = F'(0,0) = \begin{pmatrix} h_y(0,0) & h_z(0,0) \\ g_y(0,0) & g_z(0,0) \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{7}$$

The proof of Theorem 3 will be based on checking conditions 1–3 of Theorem 2 for the reduced dynamics of the DAE (4) on the solution manifold (4b). Conditions 1 and 2 will be derived in a more or less straightforward manner; condition 3 is not trivial, though. Remember that the goal is to state the conditions in terms of the original problem setting, that is, in terms of h and g (that is, of F), as it is done in our statement of condition 3. But note that it is the matrix pencil $\lambda E - F'(0,0)$ (and not the matrix $F'(0,0)$) that is assumed to have a double, index-two zero eigenvalue; this means that it is not obvious that there should exist a vector $\overline{q}$ satisfying the requirement depicted in (5) in light of the previous hypotheses. As a cautionary example, consider, for instance, the Schur reduction of

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad (M/D) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and note that $\lambda = 0$ is a double eigenvalue for (M/D) but a simple one for M; no generalized eigenvector exists in this case for M. This cannot occur in the setting of Theorem 3 (that is, there will indeed exist a $\overline{q}$ satisfying the condition in (5)) because of item (c) of Lemma 3 below.

LEMMA 3. *Given M and (M/D) in (6), consider the operators L and $T : \mathbb{R}^r \to \mathbb{R}^{r+p}$ defined by*

$$Lu = \begin{pmatrix} u \\ -D^{-1}Cu \end{pmatrix}, \quad Tu = \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

Then

(a) $\overline{p} \in \ker M \Leftrightarrow \overline{p} = Lp$, *with* $p \in \ker (M/D)$;

(b) $w \in \operatorname{im}(M/D) \Leftrightarrow Tw \in \operatorname{im} M$. *Actually,* $Tw = M\overline{u} \Leftrightarrow \overline{u} = Lu$, *with* $w = (M/D)u$;

(c) *if* $\ker M \subseteq \operatorname{im} T = \mathbb{R}^r \times \{0\}$, *then* $\overline{q} \in \ker M^2 \Leftrightarrow \overline{q} = Lq$, *with* $q \in \ker (M/D)^2$.

Proof. Items (a) and (b) are immediate in light of the definition of (M/D) in (6). Regarding (c), let $\overline{q} \in \ker M^2$ and denote $\overline{w} = M\overline{q} \in \ker M \cap \operatorname{im} M$. Then

(i) since $\overline{w} \in \ker M$, then $\overline{w} = Lw$ with $w \in \ker (M/D)$ because of (a);

(ii) owing to the hypothesis $\ker M \subseteq \operatorname{im} T$, necessarily $\overline{w} = Tu$ for a certain u; additionally, because of the form of L and T it follows that $u = w$ and then $\overline{w} = Lw = Tw$;

(iii) the condition $\overline{w} = Tw = M\overline{q}$ implies, because of (b), that $\overline{q} = Lq$, with $w = (M/D)q$;

(iv) finally, since $w \in \ker (M/D)$ (cf. (i)), it follows that $q \in \ker (M/D)^2$.

The converse result in (c) is entirely analogous and the proof is left to the reader.□

With M as defined in (7), the linear operator L is the differential at the origin of the parameterization $y \to (y, \psi(y))$ of the manifold $\mathcal{M}$ in (4b), with ψ given by the implicit function theorem, and therefore L will define an isomorphism $\mathbb{R}^r \to \ker \begin{pmatrix} C & D \end{pmatrix} = T_{(0,0)}\mathcal{M}$. Item (a) in Lemma 3 expresses that L also induces an isomorphism between the spaces $\ker (M/D) \to \ker M \subseteq T_{(0,0)}\mathcal{M}$. Moreover, in the scenario assumed in (c), one can check that $\ker M^2 \subseteq T_{(0,0)}\mathcal{M}$ and that L also induces an isomorphism $\ker (M/D)^2 \to \ker M^2$.

For later use, we also note that a coordinate description α of a map β defined on $\mathcal{M}$, that is, a relation of the form $\alpha(y) = \beta(y, \psi(y))$, implies $\alpha'(0) = \beta'(0, \psi(0))L$. We will make use of this remark in the final step of the proof of Theorem 3.

Proof of Theorem 3. 1. The hypothesis that $g_z(0,0)$ is nonsingular implies, by the implicit function theorem, that $\mathcal{M} \equiv g = 0$ is locally a manifold that can be described by $z = \psi(y)$. The goal is then to apply Theorem 2 to the reduced system

$$y' = f(y) = h(y, \psi(y)). \tag{8}$$

Specifically, we need to check that the requirements imposed on $F = (h, g)$ yield the conditions 1–3 in Theorem 2.

2. The first condition holds trivially, since

$$f(y_1, 0) = h(y_1, 0, \psi(y_1, 0)) = h(y_1, 0, 0) = 0,$$

where the second identity is due to the fact that $\psi(y_1, 0) = 0$ because $g(y_1, 0, 0) = 0$.

3. The second condition also follows easily from the implicit function theorem, since the linearization of the reduced system (8) at the origin is

$$f'(0) = h_y(0,0) + h_z(0,0)\psi'(0) = h_y(0,0) - h_z(0,0)(g_z(0,0))^{-1} g_y(0,0),$$

and the spectrum of $f'(0) = A - BD^{-1}C$ equals that of the matrix pencil $\lambda E - M$, as an immediate consequence of Schur's lemma:

$$\begin{aligned} \det(\lambda E - M) &= \det\left(\lambda \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \\ &= \det \begin{pmatrix} \lambda I_r - A & -B \\ -C & -D \end{pmatrix} = \det\left(\lambda I_r - (A - BD^{-1}C)\right)\det(-D). \end{aligned}$$

4. The only nonimmediate step consists in checking that

$$F''(0,0)\overline{p}\overline{q} \notin \operatorname{im} F'(0,0),$$

where $\overline{p} \in \ker F'(0,0) - \{0\}$, $\overline{q} \in \ker (F'(0,0))^2 - \ker F'(0,0)$ implies

$$f''(0)pq \notin \operatorname{im} f'(0),$$

with $p \in \ker f'(0) - \{0\}$ and $q \in \ker (f'(0))^2 - \ker f'(0)$. Note that item (c) of Lemma 3 applies because $\ker F'(0,0) = \operatorname{span}\{e_1\}$, and then $\overline{q}$ yields a generalized eigenvector q of $f'(0)$, with $\overline{q} = Lq$.

It then suffices to use the characterization

$$F''(0,0)\overline{p}\overline{q} \notin \operatorname{im} F'(0,0) \Leftrightarrow (\det F')'(0,0)\overline{q} \neq 0$$

following from Lemma 1 because, differentiating $\det F' = \det g_z \det(F'/g_z)$ and using the fact that $\det(F'/g_z)(0,0) = \det(f')(0) = 0$ (because $\lambda = 0$ is an eigenvalue), we have

$$(\det F')'(0,0) = \det g_z(0,0)(\det(F'/g_z))'(0,0);$$

additionally, since $\det g_z(0,0) \neq 0$, it follows that

$$(\det F')'(0,0)\overline{q} \neq 0 \Leftrightarrow (\det(F'/g_z))'(0,0)Lq \neq 0 \Leftrightarrow (\det(f'))'(0)q \neq 0,$$

because $\det f'(y) = \det(F'/g_z)(y, \psi(y)) \Rightarrow (\det(f'))'(0) = (\det(F'/g_z))'(0,0)L$ as indicated right before the proof of Theorem 3. Finally,

$$(\det(f'))'(0)q \neq 0 \Leftrightarrow f''(0)pq \notin \operatorname{im} f'(0),$$

and the proof is complete. ∎

Remark. As in the explicit ODE case, condition 3 can be recast as

$$F_{x_1 x}(0,0)\overline{q} \notin \operatorname{im} F'(0,0),$$

where $x = (y, z)$ and $F_{x_1 x}$ stands for the matrix of partial derivatives $(\frac{\partial^2 F_i}{\partial x_1 \partial x_j})$.

4. TBWP in memristive circuits. The memristor (an abbreviation for *memory-resistor*) is a new electronic device governed by a nonlinear, C^1 flux-charge relation of the form $\varphi = \phi(q)$. The existence of such a device was predicted for symmetry reasons by Chua in 1971 [11], but it was not until 2008 that it began to attract broad attention. The reason for this was the report in [53] of the design of a nanometric memristor by the HP company. The key aspect of this device is that, by differentiation of the aforementioned constitutive relation, one gets the voltage-current relation

$$v = M(q)i, \tag{9}$$

with $M(q) = \phi'(q)$. For later use, we will assume that ϕ is a C^2 map. Note that in (9) the "resistance" (or, better, *memristance*) M depends on $q(t) = \int_{-\infty}^{t} i(\tau)d\tau$, so that the device somehow keeps track of its own history (hence the memory-resistor name). A great deal of research has been directed to this and other related devices since 2008; cf. [1, 17, 24, 28, 29, 33, 35, 36, 41, 43, 45, 55] as a sample of the literature.

It is easy to check that memristors systematically yield manifolds of nonisolated equilibrium points; find details below. In order to focus on problems with *lines* of equilibria, we will restrict our attention to circuits including a single memristor, besides capacitors, inductors, resistors, and (independent) voltage and current sources. Capacitors, inductors, and resistors may be nonlinear, and they will respectively be assumed to be defined by a voltage-dependent capacitance matrix $C(v_c)$, a current-dependent inductance matrix $L(i_l)$, and a current-controlled C^1 description $v_r = \gamma(i_r)$ in the case of resistors; for later use we denote the resistance matrix $\gamma'(i_r)$ as $R(i_r)$. All three matrices need not be diagonal, allowing for the presence of coupling effects in the corresponding sets of devices. Capacitors, inductors, and resistors are said to be strictly locally passive at a given operating point if the corresponding characteristic matrix, that is, $C(v_c)$, $L(i_l)$, or $R(i_r)$, is positive definite (a square matrix P is positive definite if it verifies $v^{\mathsf{T}}Pv > 0$ for any nonvanishing real vector v; note that we do not require these matrices to be symmetric). We assume the circuit to be autonomous, that is, sources are DC ones, or, in mathematical terms, they take constant values (grouped together in two vectors V and I).

4.1. Circuit modeling based on differential-algebraic equations. In the setting defined above, the circuit equations can be modeled by the differential-algebraic system (see, e.g., [42])

$$q_m' = i_m, \tag{10a}$$
$$C(v_c)v_c' = i_c, \tag{10b}$$
$$L(i_l)i_l' = v_l, \tag{10c}$$
$$0 = B_m M(q_m)i_m + B_c v_c + B_l v_l + B_r \gamma(i_r) + B_u V + B_j v_j, \tag{10d}$$
$$0 = Q_m i_m + Q_c i_c + Q_l i_l + Q_r i_r + Q_u i_u + Q_j I, \tag{10e}$$

where the subscripts m, c, l, r, u, j are used for memristors, capacitors, inductors, resistors, voltage sources, and current sources, respectively. It is worth emphasizing that (10d) and (10e) express Kirchhoff voltage and current laws in terms of the so-called loop and cutset matrices B and Q (cf. the appendix and [7, 42, 56]); we split these matrices as $B = (B_m\ B_c\ B_l\ B_r\ B_u\ B_j)$ and $Q = (Q_m\ Q_c\ Q_l\ Q_r\ Q_u\ Q_j)$, where B_m (resp., B_c, B_l, B_r, B_u, B_j) corresponds to the columns of B accommodating memristors (resp., capacitors, inductors, resistors, voltage sources, current sources), and the same notational convention applies to the cutset matrix. For later use, note that by denoting $y = (q_m, v_c, i_l)$, $z = (i_m, i_c, v_l, i_r, v_j, i_u)$, the DAE (10) takes the form

$$E(y)y' = h(y, z), \tag{11a}$$

$$0 = g(y, z). \tag{11b}$$

A brief digression regarding DAE-based (versus state-space) circuit modeling is worthwhile at this point. On the one hand, a state-space (explicit ODE) reduction of a DAE circuit model does not exist in the presence of so-called *singular points*, yielding generically impasse phenomena. This can be traced back at least to the analysis of jump phenomena in [48], related to the presence of relaxation oscillations and canards; see also [13, 16, 51]. From a physical point of view, these singularities are sometimes seen as the result of a defective modeling process, with small (regular or singular) perturbations leading to a regularization of the model which avoids these phenomena; see [14, 25, 50] in this regard.

On the other hand, even if such a state-space description exists, either in the absence of singularities or as the result of a regularization process, its derivation may be hardly automatable or could require more assumptions than necessary from a qualitative or a numerical point of view. The differential-algebraic or *semistate* approach stems from this fact; besides the seminal work of Newcomb [37], worth mentioning in this approach are the papers by Chua and co-workers on constrained systems and generalized vector fields in the late 1980s [13, 14] and, more recently, the works [18, 23, 26, 42, 54, 56, 57], among others. Avoiding the need for such a state reduction may be of analytical interest by itself for different purposes, but in order to get actually useful tools we need mathematical statements which can be directly applied to DAE models; Theorem 3 above is a result in this spirit. Moreover, keeping both Kirchhoff laws and the devices' constitutive relations explicitly in their original forms (as in the DAE circuit model (10)) paves the way for a topological approach to the bifurcation phenomenon here considered. Details are given in the following sections.

4.2. Equilibria and transcritical bifurcations without parameters in memristive circuits. Equilibria of (11) are defined by the pair of conditions $h(y, z) = 0$, $g(y, z) = 0$, that is,

$$i_m = i_c = v_l = 0, \tag{12a}$$

$$B_c v_c + B_r \gamma(i_r) + B_u V + B_j v_j = 0, \tag{12b}$$

$$Q_l i_l + Q_r i_r + Q_u i_u + Q_j I = 0. \tag{12c}$$

Note that the variable q_m is not at all involved in (12). This means that, necessarily, no equilibrium point may be isolated, since the variable q_m unfolds any given equilibrium point to a line (or even a higher dimensional set) of equilibria. In our working setting, equilibria will actually define a line, as a consequence of the condition $\operatorname{cork} F' = 1$ shown within the proof of Theorem 4.

The previous remarks drive the stability analysis of equilibria in memristive circuits to the mathematical context considered in [4, 19, 20, 21, 31]. In this setting, the existence of an m-dimensional manifold of equilibria implies that at least m eigenvalues of the linearization of the vector field at any of these equilibria are null. The manifold is then said to be *normally hyperbolic* (locally around such an equilibrium) if the remaining eigenvalues are not in the imaginary axis. The failing of the normal hyperbolicity requirement typically yields a bifurcation without parameters [19, 20, 21, 31], where the qualitative properties of the local phase portrait change.

In [45] one can find graph-theoretic conditions under which any manifold of equilibria of a strictly locally passive memristive circuit is guaranteed to be normally hyperbolic. It is therefore natural to address what happens when the passivity assumption does not hold; allowed by Theorem 3, Theorem 4 below answers this question, again in circuit-theoretic terms, for circuits with one memristor which becomes locally active at a given operating point, yielding a transcritical bifurcation without parameters. Note that in the statement of Theorem 4, a VMC-loop is a loop composed only of voltage sources, memristors, and/or capacitors. ILC-cutsets, VML-loops, etc. are defined analogously.

THEOREM 4 (TBWP in memristive circuits). *Consider a nonlinear circuit with a single memristor, modeled by* (10). *Fix an equilibrium point* $(q_m^*, v_c^*, i_l^*, i_m^*, i_c^*, v_l^*, i_r^*, v_j^*, i_u^*)$ *(with* $i_m^* = 0,\ i_c^* = 0,\ v_l^* = 0$*), and assume that the following conditions hold.*

1. *The circuit displays neither VMC-loops nor ILC-cutsets.*
2. *There is a unique VML-loop, which includes the memristor and at least one inductor.*
3. *The capacitance, inductance, and resistance matrices* $C(v_c^*)$, $L(i_l^*)$, $R(i_r^*) = \gamma'(i_r^*)$ *are positive definite, with* $C(v_c^*)$ *and* $L(i_l^*)$ *symmetric; additionally,* $M(q_m^*) = 0$ *and* $M'(q_m^*) \neq 0$.

Then the circuit undergoes a transcritical bifurcation without parameters at the aforementioned equilibrium point; moreover, near this bifurcating equilibrium, all eigenvalues of the linearization (but the null one) have negative real part in the region where $M(q_m) > 0$*, whereas a single (real) eigenvalue becomes positive at points where* $M(q_m) < 0$.

In the proof of Theorem 4 we will make use of some graph-theoretic results which are compiled in advance. Proofs of these auxiliary results can be found in [3, 7, 42, 45].

LEMMA 4. *Let* B_i *and* Q_i *denote, for* $i = 1,\ 2,\ 3$*, the submatrices of* B *and* Q *defined by three pairwise-disjoint branch sets* K_1, K_2, K_3 *of a given directed graph. If* P *is a positive definite matrix, then*

$$\ker \begin{pmatrix} B_1 & 0 & B_3 P \\ 0 & Q_2 & Q_3 \end{pmatrix} = \ker B_1 \times \ker Q_2 \times \{0\}.$$

The same terminological convention is used in Lemma 5 below. By a K_i-cutset (resp., loop) we mean a cutset (resp., loop) defined only by branches belonging to K_i.

LEMMA 5. *The identity* $\ker B_1 = \{0\}$ *(resp.,* $\ker Q_2 = \{0\}$*) holds if and only if the digraph has no* K_1*-cutsets (resp.,* K_2*-loops).*

The proof of Theorem 4 below is based on Theorem 3, which for simplicity was stated under the assumption that the bifurcating equilibrium is located at the origin. Obviously, we can make use of this result at a generic equilibrium (y^*, z^*) (with $y^* = (q_m^*, v_c^*, i_l^*)$ and $z^* = (i_m^*, i_c^*, v_l^*, i_r^*, v_j^*, i_u^*)$), and we will do so without further explicit mention.

Proof of Theorem 4. Note first that the strict passivity assumption on $C(v_c^*)$ and $L(i_l^*)$ makes these matrices nonsingular, and therefore the maps h and g from (4) have (at least locally) the form

$$h(y,z) = \begin{pmatrix} i_m \\ (C(v_c))^{-1} i_c \\ (L(i_l))^{-1} v_l \end{pmatrix},$$

$$g(y,z) = \begin{pmatrix} B_m M(q_m) i_m + B_c v_c + B_l v_l + B_r \gamma(i_r) + B_u V + B_j v_j \\ Q_m i_m + Q_c i_c + Q_l i_l + Q_r i_r + Q_u i_u + Q_j I \end{pmatrix},$$

where we are denoting $y = (q_m, v_c, i_l),\ z = (i_m, i_c, v_l, i_r, v_j, i_u)$.

1. The matrix of partial derivatives $g_z(y^*, z^*)$ is (using $i_m^* = 0,\ M(q_m^*) = 0$)

$$g_z(y^*, z^*) = \begin{pmatrix} 0 & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix},$$

which is invertible in light of Lemmas 4 and 5, since $R(i_r^*)$ is positive definite and there are neither IL-cutsets (which are a particular instance of an ILC-cutset) nor VMC-loops.

2. Denoting $y = (q_m, \tilde{y})$, the conditions $h(q_m, \tilde{y}^*, z^*) = 0$, $g(q_m, \tilde{y}^*, z^*) = 0$ (arising in condition 1 of Theorem 3) are trivially met (cf. (12)).

3. Condition 2 of Theorem 3 involves a matrix pencil spectrum which is given by the determinant of

(13)
$$\begin{pmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda I_c & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda I_l & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ 0 & -B_c & 0 & 0 & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & -Q_u \end{pmatrix},$$

and which can be written as $\lambda d(\lambda)$ with $d(\lambda) = \det K(\lambda)$, provided that $K(\lambda)$ is the bottom-right submatrix of (14) (that is, the one obtained after removing the top row and the left column).

The eigenvalue $\lambda = 0$ being a double one amounts to $d(0) = 0$ (i.e., to $K(0)$ being singular) with $d'(0) \neq 0$. Additionally, provided that $\operatorname{cork} K(0) = 1$ (a condition which will be proved to hold), then the condition $d'(0) \neq 0$ is equivalent (cf. Lemma 1) to $K'(0)p \notin \operatorname{im} K(0)$ for $p \in \ker K(0) - \{0\}$. Finally, for the (double) zero eigenvalue to be index-two it will be enough to show that $\operatorname{cork} F'(y^*, z^*) = 1$. We examine this set of conditions in items 4, 5, and 6 below.

4. The matrix $K(0)$ reads as

$$\begin{pmatrix} 0 & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ -B_c & 0 & 0 & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & -Q_u \end{pmatrix}$$

and, following Schur's lemma (Lemma 2), this matrix is easily seen to have the same corank as

$$\begin{pmatrix} B_c & 0 & 0 & B_r R(i_r^*) & B_j & 0 \\ 0 & Q_l & Q_m & Q_r & 0 & Q_u \end{pmatrix}.$$

From Lemmas 4 and 5, the positive definiteness of $R(i_r^*)$, the absence of IC-cutsets, and the existence of a unique VML-loop, it follows that $\operatorname{cork} K(0) = 1$, with

$$\ker K(0) = \operatorname{span}\{(0, p_l, p_m, 0, 0, 0, 0, p_u)\},$$

where $(p_l, p_m, p_u) \in \ker \begin{pmatrix} Q_l & Q_m & Q_u \end{pmatrix}$ (and note, for later use, that $p_l \neq 0$).

5. The condition $K'(0)p \notin \operatorname{im} K(0)$ is

$$\begin{pmatrix} 0 \\ p_l \\ 0 \\ 0 \end{pmatrix} \notin \operatorname{im} \begin{pmatrix} 0 & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ -B_c & 0 & 0 & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & Q_u \end{pmatrix}.$$

Assuming that this condition does not hold, we would have a solution for

$$B_c u_1 - B_l L(i_l^*) p_l + B_r R(i_r^*) u_6 + B_j u_7 = 0.$$

Together with the identity $Q_l p_l + Q_m p_m + Q_u p_u = 0$ and the orthogonality of the so-called cut and cycle spaces $\ker B$, $\ker Q$ (cf. [7]), we would get $p_l^{\mathsf{T}} L(i_l^*) p_l = 0$ and therefore $p_l = 0$ (since $L(i_l^*)$ is positive definite), against the fact that $p_l \neq 0$.

Hence, $K'(0)p \notin \operatorname{im} K(0)$ for $p \in \ker K(0) - \{0\}$, and as indicated above this implies that $\lambda = 0$ is a double eigenvalue.

6. The matrix $F'(y^*, z^*)$ is

$$F'(y^*, z^*) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (L(i_l^*))^{-1} & 0 & 0 & 0 \\ 0 & B_c & 0 & 0 & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix}$$

and, via Schur's lemma, $\operatorname{cork} F'(y^*, z^*) = 1$ follows again from Lemmas 4 and 5, which imply that

$$\ker \begin{pmatrix} B_c & 0 & B_r R(i_r^*) & B_j & 0 \\ 0 & Q_l & Q_r & 0 & Q_u \end{pmatrix} = \{0\}.$$

As indicated above, this implies that the double zero eigenvalue is indeed index two.

7. The fact that all nonvanishing eigenvalues ($\lambda \neq 0$) have nonzero real part follows from the eigenvalue-eigenvector equations of the pencil, which can be written as

$$B_c u_c + \lambda B_l L(i_l^*) w_l + B_r R(i_r^*) w_r + B_j u_j = 0,$$
$$\lambda Q_c C(v_c^*) u_c + Q_l w_l + Q_r w_r + Q_m w_m + Q_u w_u = 0$$

together with $\lambda \sigma_m = w_m$, $w_c = \lambda C(v_c^*) u_c$, $u_l = \lambda L(i_l^*) w_l$. By taking conjugate transposes and using the orthogonality of the cut and cycle spaces, we derive

$$(\mathrm{Re}\,\lambda)\,(u_c^\star C(v_c^*) u_c + w_l^\star L(i_l^*) w_l) + w_r^\star \frac{R(i_r^*) + (R(i_r^*))^{\mathsf{T}}}{2} w_r = 0. \tag{14}$$

Now, if $\mathrm{Re}\,\lambda = 0$, the positive definiteness of $R(i_r^*)$ implies $w_r = 0$, and then

$$B_c u_c + \lambda B_l L w_l + B_j u_j = 0, \tag{15a}$$
$$\lambda Q_c C u_c + Q_l w_l + Q_m w_m + Q_u w_u = 0. \tag{15b}$$

But in this setting the hypothesis that there are no ILC-cutsets implies, in light of (15a), $u_c = 0$ (and then $w_c = 0$), $w_l = 0$ (so that $u_l = 0$), and $u_j = 0$. Additionally, the absence of VM-loops and (15b) would then imply $w_m = 0$ (and then $\sigma_m = 0$) and $w_u = 0$; this would yield a vanishing eigenvector, which is a contradiction in terms.

8. In order to check that condition 3 of Theorem 3 holds, we take $\overline{q}$ from the requirement $F'(y^*, z^*)\overline{q} \in \ker F'(y^*, z^*) - \{0\}$, which gives $\overline{q}$ the form

$$\overline{q} = (\overline{q}_1, 0, \overline{q}_3, \overline{q}_4, 0, 0, 0, 0, \overline{q}_9), \text{ with } (\overline{q}_3, \overline{q}_4, \overline{q}_9) \in \ker \begin{pmatrix} Q_l & Q_m & Q_u \end{pmatrix} - \{0\}$$

and, in particular, $\overline{q}_4 \neq 0$. The condition $F''(y^*, z^*)\overline{pq} \notin \operatorname{im} F'(y^*, z^*)$ then reads as

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ B_m M'(q^*)\overline{q}_4 \\ 0 \end{pmatrix} \notin \operatorname{im} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (L(i_l^*))^{-1} & 0 & 0 & 0 \\ 0 & B_c & 0 & 0 & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix}.$$

Again, assuming that this condition is not met, we would derive the existence of a nontrivial solution for

$$B_c u_2 - B_m M'(q^*)\overline{q}_4 + B_r R(i_r^*) u_7 + B_i u_8 = 0,$$

but the presence of a VML-loop with the memristor rules out, because of the colored branch theorem (according to which, in a three-color graph with just one blue branch, this branch cannot form simultaneously a loop exclusively with green branches and a cutset only with red branches; cf. [34, 58]), the existence of any CMRI-cutset including the memristor. This would yield $M'(q^*)\overline{q}_4 = 0$, against the fact that $M'(q^*) \neq 0 \neq \overline{q}_4$.

9. Finally, the proof that all eigenvalues (but the null one) have negative real part at points of the equilibrium line where $M(q_m) > 0$ is essentially similar to the one in item 7. Note only that the fact that $M(q_m) \neq 0$ makes the eigenvalue-eigenvector equations read

$$\begin{aligned} B_c u_c + \lambda B_l L(i_l) w_l + B_r R(i_r) w_r + B_m M(q_m) w_m + B_j u_j &= 0, \\ \lambda Q_c C(v_c) u_c + Q_l w_l + Q_r w_r + Q_m w_m + Q_u w_u &= 0, \end{aligned}$$

and (14) is now

$$(\operatorname{Re}\lambda)\,(u_c^\star C(v_c) u_c + w_l^\star L(i_l) w_l) + w_r^\star \frac{R(i_r) + (R(i_r))^\mathsf{T}}{2} w_r + w_m^\star M(q_m) w_m = 0.$$

But since not only $C(v_c)$, $L(i_l)$, and $R(i_r)$ but also $M(q_m)$ are positive definite, from the assumption $\operatorname{Re}\lambda \geq 0$ we would derive $w_m = w_r = 0$, and the reasoning proceeds as in item 7 above to show that all nonvanishing eigenvalues must verify $\operatorname{Re}\lambda < 0$. On the other hand, the fact that only one real eigenvalue changes sign (and hence becomes positive) in the transition to the region where $M(q_m) < 0$ follows (locally) from the TBWP phenomenon itself. This completes the proof. ∎

Note that Theorem 4 assumes that, except for the memristor, the remaining circuit devices are strictly locally passive. If this assumption is relaxed, things become more complicated; we present some results in the nonpassive context in section 5.

4.3. Topological remarks. The topological conditions arising in Theorem 4 are worthy of discussion. Essentially, there are three different groups of conditions, which are somehow merged into the statement of the theorem. First, there is a group of (say, index-one) conditions which, in a different language, can be traced back to the classical works [6, 9], according to which VC-loops and IL-cutsets must be excluded in order to avoid constraints among the dynamic circuit variables (typically, capacitor voltages and inductor currents); in the DAE setting, such so-called *degenerate* configurations yield index-two models [18, 42, 56]. In our setting (cf. item 1 of the proof above) IL-cutsets are implicitly ruled out by the absence of ILC-cutsets in condition 1 of Theorem 4; on the other hand, since the memristor is charge-controlled and the memristance M does vanish at the bifurcation point, we need to exclude VMC-loops which would yield a singularity when M vanishes. It is reasonable to expect that, with the appropriate transversality assumptions and precluding certain additional configurations, a VMC-loop would systematically result in a hypersurface of impasse points defined by the vanishing of the memristance; an example is discussed in detail in [16].

The second group of conditions is related to the existence of null eigenvalues. Again, it is known in classical circuit theory that the presence of either VL-loops or IC-cutsets is responsible for a null eigenvalue in the linearized dynamics [32]. In our framework, IC-cutsets are again precluded by the absence of ILC-cutsets; in turn, the assumed existence of a VML-loop (which may be in particular an ML-loop) with the memristor and at least one inductor results in a zero eigenvalue at the bifurcation point, which is in turn responsible for the transcritical bifurcation without parameters. This occurs if the memristance $M(q_m)$ vanishes at a given q_m^* and eventually becomes negative beyond that point (see also, in this regard, the examples discussed in sections 5 and 6). A simple parallel connection of a memristor and a linear inductor yields this phenomenon, as shown in [22]. These conditions are used explicitly in item 4 of the proof of Theorem 4.

Finally, the third group of topological conditions is related to the existence of purely imaginary eigenvalues (PIEs). In this case (cf. [47]) both a VCL-loop and an ILC-cutset are required (in a passive setting) for a PIE to exist; essentially, as detailed in item 7 of the proof, in our context the assumed absence of ILC-cutsets is enough to rule out eigenvalues in the imaginary axis.

5. Nonpassive problems. Consider the circuit displayed in Figure 1. Note that the absence of a (V)ML-loop rules out a direct application of Theorem 4 in order to characterize an eventual transcritical bifurcation without parameters in this circuit. However, it is easy to check that the series connection of the linear resistor R and the memristor M can itself be modeled as a memristor with memristance $R + M(q)$. Moreover, provided that $R + M(q^*) = 0$ at a given q^* (with $M'(q^*) \neq 0$), the circuit is expected and can be easily shown to undergo a TBWP. This indicates that the existence of a VML-loop is not necessary for the bifurcation to occur. Actually, both conditions $M(q^*) = 0$ in Theorem 4 and $R + M(q^*) = 0$ are particular (and simple) examples of a more general bifurcating condition derived below (condition 3 in Proposition 1; see also the remark below and the comments concerning Figure 2).

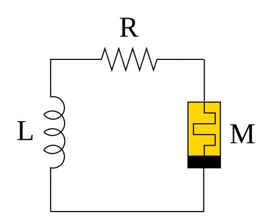

FIG. 1. *MRL-circuit.*

Obviously, for the condition $R + M(q^*) = 0$ to hold, either both R and M must vanish or one of them needs to become negative. In particular, when R becomes zero or negative, the problem does not fit the strict passivity assumption for resistors stated in Theorem 4. This means that a wider framework is needed to address this phenomenon in general. Although in its broad generality this is a difficult problem, some remarks in this direction can be provided, as detailed in what follows.

5.1. Null eigenvalues in the nonpassive setting. We provide below conditions guaranteeing, in nonpassive circuits with a single memristor, that the null eigenvalue is indeed a multiple one, generalizing (as detailed later; cf. the remark after Proposition 1) the framework considered in Theorem 4. We use the notion of both a proper tree and an L-proper tree. Split the branches of a given connected graph $\mathcal{G}$ into three pairwise disjoint sets K_1, K_2, and K_3 in such a way that K_1 includes no loops and K_3 no cutsets. Then, as a consequence of the matroid structure of the set of acyclic subgraphs of $\mathcal{G}$ [38], one can guarantee that there exists at least one spanning tree including all branches from K_1 and none from K_3 (an explicit proof can be found, e.g., in [8]). Such a spanning tree is called (in general) a *proper tree*. In circuit theory, this term is usually restricted to connected circuits without VC-loops and IL-cutsets, to denote a spanning tree including all voltage sources and capacitors, and neither current sources nor inductors. This notion can be traced back at least to [6]. We will also make use of the (in a certain sense dual) concept of an *L-proper tree*, which is a spanning tree including all voltage sources and inductors, and neither current sources nor capacitors; such a tree exists if and only if the circuit has neither VL-loops nor IC-cutsets. From [46] we borrow the concept of an MR-product; given a spanning tree in a connected circuit with voltage and current sources, capacitors, inductors, resistors, and memristors, the *MR-product* of this tree is simply the product of all resistances and memristances in the co-tree branches (namely, the branches that do not belong to the spanning tree), evaluated at equilibrium and setting this product to 1 if all resistors and memristors are actually located in the tree.

PROPOSITION 1. *Consider, as in Theorem* 4*, a circuit with a single memristor displaying an equilibrium point at a given* (y^*, z^*)*, with* $y^* = (q_m^*, v_c^*, i_l^*)$ *and* $z^* = (i_m^*, i_c^*, v_l^*, i_r^*, v_j^*, i_u^*)$*. Assume that* $C(v_c^*)$ *and* $L(i_l^*)$ *are nonsingular, in addition to the following.*

1. *The circuit displays no VC-loops, IL-cutsets, VL-loops, or IC-cutsets.*
2. *The sum of MR-products in proper trees does not vanish.*
3. *The sum of MR-products in L-proper trees does vanish.*

Then the algebraic multiplicity of the zero eigenvalue at (y^*, z^*) *is greater than one.*

The proof of this result follows from the results detailed in [46]. First, the absence of VC-loops and IL-cutsets, together with the nonvanishing condition on the MR-product sum in proper trees, guarantees the matrix $g_z(y^*, z^*)$ to be nonsingular; cf. Proposition 3 in [46]. Note that this matrix has now the form

$$g_z(y^*, z^*) = \begin{pmatrix} B_m M(q_m^*) & 0 & B_l & B_r R(i_r^*) & B_j & 0 \\ Q_m & Q_c & 0 & Q_r & 0 & Q_u \end{pmatrix}. \tag{16}$$

Additionally, the null eigenvalue having a multiplicity greater than one is in a way the dual property to the one above and relies on the structure of the matrix $K(0)$

from (14), which in the presence of a possibly nonvanishing memristance reads as

$$\begin{pmatrix} 0 & 0 & 0 & -(C(v_c^*))^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(L(i_l^*))^{-1} & 0 & 0 & 0 \\ -B_c & 0 & -B_m M(q_m^*) & 0 & -B_l & -B_r R(i_r^*) & -B_j & 0 \\ 0 & -Q_l & -Q_m & -Q_c & 0 & -Q_r & 0 & -Q_u \end{pmatrix}.$$

Again, this matrix can be checked to be singular if and only if it is the matrix

$$\begin{pmatrix} B_c & 0 & B_m M(q_m^*) & B_r R(i_r^*) & B_j & 0 \\ 0 & Q_l & Q_m & Q_r & 0 & Q_u \end{pmatrix}, \tag{17}$$

but from Proposition 3 in [46] one can show that, in the absence of VL-loops and IC-cutsets, the latter matrix is singular if and only if the sum of MR-products in L-proper trees does vanish. Details are not difficult and are left to the reader. The above-referenced duality property becomes clear in light of the matrices (16) and (17), which have exactly the same form after a change of reactive devices (capacitors and inductors) and an obvious column reordering.

Proposition 1 opens a way for future research, which should address the remaining conditions from Theorem 3 in order to characterize the TBWP in this wider setting. Note that Proposition 1 does not require any passivity assumption on the circuit characteristic matrices.

Remark. In the presence of a VML-loop, as in the setting of Theorem 4, the memristor must by definition belong to the co-tree of all L-proper trees (since these must accommodate all voltage sources and inductors); M is therefore a common factor in all MR-products, and therefore the condition $M = 0$ in Theorem 4 arises naturally.

Proposition 1 above explains, in topological terms, why the bifurcating condition in the circuit of Figure 1 is $M(q_m^*) + R = 0$. Just note that the circuit has two L-proper trees, displayed in Figure 2, the co-trees of which amount, respectively, to the memristor M and the resistor R; the sum of products arising in item 3 of Proposition 1 is just $M(q_m^*) + R$, and for this reason the vanishing of this sum is responsible for a multiple zero eigenvalue supporting the TBWP. However, the scope of this result goes much further than this elementary (pedagogic) example, since the computation of spanning trees is an easily automatable task, and therefore the result applies to much more complex circuits. A case of intermediate complexity is discussed in subsection 5.2 for illustrative purposes.

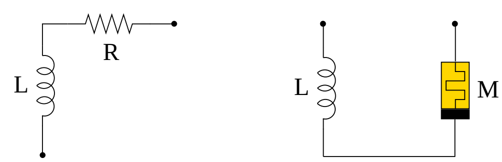

FIG. 2. *L-proper trees in the circuit of Figure* 1.

5.2. A memristive artificial neural network. Memristors provide an excellent framework for the implementation of artificial neural networks. The key reason is that they define a nanometric scale, electrically adaptable device perfectly suited for implementing neural synapses, emulating (to a certain extent) the Spike-Timing-Dependent Plasticity (STDP) mechanism from biological neural systems [49]. A lot of recent literature explores this idea; see, e.g., [2, 27, 29, 41, 52, 59] and references therein.

In this context, we analyze below a simplification of an additive model proposed in [59]. We ignore delays and assume that each neuron is defined by a passive RC-connection, and also that the input-output function in each neuron is just implemented by a linear passive resistor. Furthermore, we focus on a problem with just two neurons and assume that the conductivities of three out of four synaptic connections are fixed in order to focus our attention on a bifurcating memristor. This simplified model is depicted in Figure 3 (left); in Figure 3 (right) we arrange the circuit in a more convenient manner for later computations.

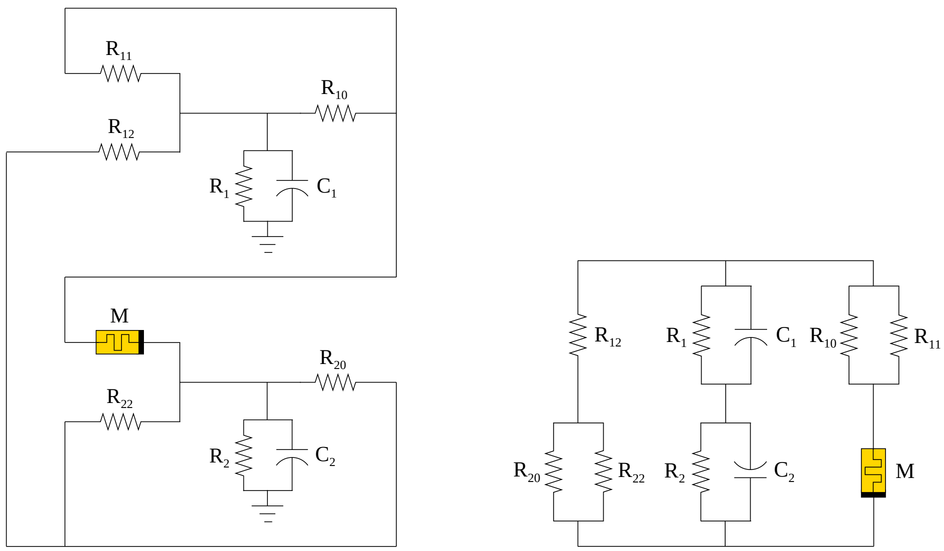

FIG. 3. *Left: Memristive network. Right: Alternative circuit description.*

Note that, for any charge q_m in the memristor and null values in the remaining circuit variables, one gets an equilibrium point. This simply expresses the presence of a line of equilibria along the q_m-coordinate axis, and the linearized dynamics exhibits a zero eigenvalue along this axis. Omitting details for the sake of brevity, a state-space model for this circuit indicates that the condition for the zero eigenvalue to be multiple is

$$R_A R_B + (R_1 + R_2)\,[(R_{20} + R_{22})R_A + (R_{10} + R_{11})R_B] = 0, \tag{18}$$

with $R_A = M(R_{10} + R_{11}) + R_{10}R_{11},\ R_B = R_{12}(R_{20} + R_{22}) + R_{20}R_{22}$. This yields the bifurcation value

$$M = \frac{-R_C - R_D}{(R_{10} + R_{11})[R_B + (R_1 + R_2)(R_{20} + R_{22})]},$$

where $R_C = R_{10}R_{11}[R_B + (R_1 + R_2)(R_{20} + R_{22})],\ R_D = R_B(R_1 + R_2)(R_{10} + R_{11})$. Actually, for negative values M smaller than the one above, and provided that all resistances are positive, a transition of an eigenvalue to the positive real semiaxis signals a stability loss due to a TBWP.

Our goal is to explain in topological terms the condition (18) on the resistances and the memristance, which makes this null eigenvalue a multiple one, in order to illustrate the scope of Proposition 1. Note that in large scale circuits the derivation

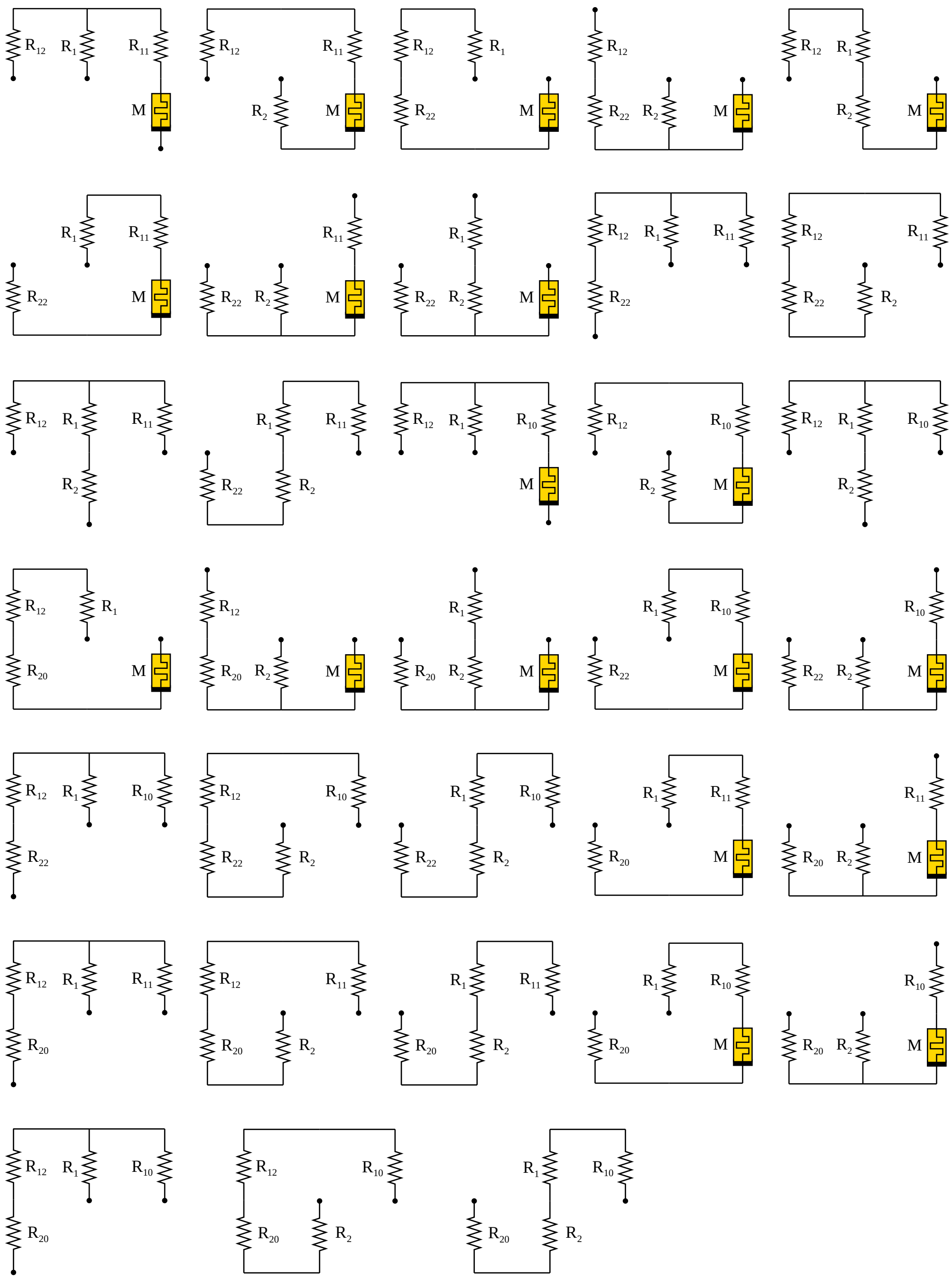

FIG. 4. *L-proper trees for the circuit in Figure* 3.

of a model and therefore the explicit computation of the bifurcation conditions are usually unfeasible, and for this reason one has no option but to resort to circuit-theoretic results such as the one in Proposition 1. To achieve this one needs to compute the set of L-proper trees of the circuit. In our present example there are actually 33 L-proper trees, depicted in Figure 4.

One can check that this set of trees actually explains the bifurcating condition (18), in light of item 3 of Proposition 1. Specifically, the terms responsible for the product $R_A R_B$ arise from the co-tree branches of the trees 5, 8, 11, 12, 15, 18, 23, 28, and 33 in Figure 4; analogously, the terms $(R_1 + R_2)(R_{20} + R_{22})R_A$ come from the co-tree branches of trees 3, 4, 9, 10, 16, 17, 21, 22, 26, 27, 31, and 32, whereas the products $(R_1 + R_2)(R_{10} + R_{11})R_B$ arise from the co-trees of the trees 1, 2, 6, 7, 13, 14, 19, 20, 24, 25, 29, and 30.

As indicated above, this graph-theoretic characterization of the bifurcation condition is easily scalable to large circuits in which the analytical (model-based) computation of such a condition is not feasible.

6. Flux-charge modeling. Some related results have been recently addressed using a different (though closely related) approach, in which a flux-charge modeling framework drives the qualitative analysis to a lower dimensional, parameterized setting [15, 16]. We briefly comment on this approach in this section. Somewhat related results have also been recently developed in order to analyze the existence of multiple attractors in memristor circuits; cf. [5, 61] and references therein.

The method proposed in [15, 16] describes the circuit dynamics on certain invariant manifolds in terms of a set of flux/charge variables in the reactive devices (capacitors and inductors). The contribution of memristors is captured via the initial values of flux and charges in these devices, which enter the dynamical description as explicit parameters. We illustrate this approach in terms of the circuit displayed in Figure 1.

To keep things as simple as possible we take $L = 1$, $R = 0$ and let the memristor be governed by the relation $\varphi_m = \phi(q_m) = -q_m + q_m^3$, with $M(q_m) = \phi'(q_m) = -1 + 3q_m^2$. Using the model (10), under the assumptions above, the equations for the circuit in Figure 1 are easily seen to amount to

$$q_m' = i_l, \tag{19a}$$

$$i_l' = -M(q_m) i_l. \tag{19b}$$

One can check that the points defined by $i_l = 0$, $q_m = \pm q_m^*$ with $q_m^* = 1/\sqrt{3}$ yield two TBWPs within the equilibrium line given by $i_l = 0$.

The approach proposed in [15, 16] obtains a description of the dynamics in invariant curves of the form $i_l + \phi(q_m) = \mu$ (with constant $\mu \in \mathbb{R}$) either as $q_l' = -\phi(q_l - q_{l0} + q_{m0}) + \phi(q_{m0}) + i_{l0}$, where q_l is the charge flowing through the inductor, or, in terms of the memristor charge q_m, as

$$q_m' = -\phi(q_m) + \phi(q_{m0}) + i_{l0}. \tag{20}$$

Note that both q_l and q_m provide a global parameterization of the aforementioned invariant curves. We focus on the description (20) for simplicity, and we emphasize that each invariant curve is uniquely identified by a value of the parameter $\mu = \phi(q_{m0}) + i_{l0}$ within (20). After some easy computations and setting $\mu^* = 2/(3\sqrt{3})$, one can check that

- for $\mu > \mu^*$, the right-hand side of (20) has a unique zero, which corresponds to a unique equilibrium in the invariant curve toward which all trajectories within this curve converge;

- for values of μ verifying $-\mu^* < \mu < \mu^*$, there are three zeros of the right-hand side of (20), which describe three equilibria (two of them stable and one unstable); and
- finally, for $\mu < -\mu^*$, the right-hand side of (20) has a unique zero, which again corresponds to a unique equilibrium in the invariant curve attracting all trajectories within the curve.

The values $\mu = \pm\mu^*$ define two separatrices, each one accommodating one of the TBWP points $\pm q_m^*$.

It is important to emphasize that the flow-invariant foliation given by the invariant curves described above is *not* transversal to the equilibrium line at the TBWP points $\pm q_m^*$ (cf. [31] in this regard, where the reader can find a detailed discussion of how such foliations, when they are transversal to the equilibrium line, unveil "hidden" parameters). In this case, even if the foliation is not transversal to the equilibrium line at the TBWP points, the local bifurcation can be recast in terms of a classical parameter; namely, the parameter $\mu = \phi(q_{m0}) + i_{l0}$ in (20) can be checked to yield (classical) saddle-node bifurcations at $\mp q_m^*$ for $\mu = \pm\mu^*$. Moreover, this reformulation in terms of a classical parameter might be systematically addressed for (at least certain families of) memristor circuits using this flux-charge formalism. This idea seems worth studying in more detail.

7. Concluding remarks. We have presented in this paper a detailed circuit-theoretic characterization of the transcritical bifurcation without parameters in circuits with one memristor, systematically yielding lines of equilibrium points. To do so, we have developed mathematical statements of the TBWP theorem for explicit ODEs in arbitrary dimension and also for semiexplicit DAEs, which are believed to be of independent interest. This allows for a graph-theoretic analysis of the bifurcation in the circuit context. Future research should provide a complete characterization of the TBWP phenomenon in nonpassive settings (along the lines discussed in section 5) and address this bifurcation in broader contexts, including, e.g., multiport networks. It would also be of interest to study systematically and in topological terms the dynamic phenomena arising from the failure of some conditions in Theorem 4, which should result in higher index DAE models, impasse phenomena, etc. Other related bifurcations, such as the Hopf bifurcation without parameters, might be analyzed in similar terms.

Appendix A. Digraph matrices. In the formulation of the circuit model (10) we make use of the so-called *loop* and *cutset* matrices defined below. Given a digraph with m edges, n nodes, and k connected components, choose an orientation in every loop and define componentwise the *loop matrix* $\tilde{B}$ as (b_{ij}), with

$$b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is in loop } i \text{ with the same orientation,} \\ -1 & \text{if edge } j \text{ is in loop } i \text{ with the opposite orientation,} \\ 0 & \text{if edge } j \text{ is not in loop } i. \end{cases}$$

This matrix has rank $m-n+k$, and a *reduced loop matrix* B is any $((m-n+k)\times m)$-submatrix of $\tilde{B}$ with full row rank.

The dual concept is that of a reduced cutset matrix. A set K of edges in a digraph is a *cutset* if the removal of K increases the number of connected components, and K is minimal with respect to this property, that is, retaining any edge from K keeps the number of components invariant. All the edges of a cutset may be shown to connect the same pair of connected components which result from the cutset deletion, and this allows one to define the orientation of a cutset, say from one of these components toward the other. This makes it possible to define the cutset matrix $\tilde{Q} = (q_{ij})$ as

$$q_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is in cutset } i \text{ with the same orientation,} \\ -1 & \text{if edge } j \text{ is in cutset } i \text{ with the opposite orientation,} \\ 0 & \text{if edge } j \text{ is not in cutset } i. \end{cases}$$

Now the rank of $\tilde{Q}$ is $n-k$, and a *reduced cutset matrix* $Q \in \mathbb{R}^{(n-k)\times m}$ is obtained by choosing any set of $n-k$ linearly independent rows of $\tilde{Q}$.

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